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Abstract

As is well known, Odd Aukrust made the axiomatisation study a field of research in national accounting. Actually, his approach to the axiomatisation was preceded by that of Ragnar Frisch. Although Frisch did not explicitly make an axiomatisation for his "ecocirc" system, his approach may be deemed to be a pioneering work of graph-theoretical type of axiomatisation of national accounting, while Aukrust's approach may be deemed to be a set-theoretical type of axiomatisation, which was followed by many researchers in this filed. Further, it is worthy of notice that his work stimulated axiomatisation of business accountings pioneered by Richard Mattessich. The work done by the two great Norwegian figures in national accounting also influenced two Japanese researchers, Itsuo Sakuma and Hiroshi Deguchi. Sakuma made a formulation by means of graph-theoretical framework following Frisch's idea and Deguchi replaced the set-theoretical approach by an algebraic approach. In this paper, after a brief survey on the axiomatisation in national as well as business accounting, the two approaches, graphical and algebraic, will be described and some remarks will be made.

Contents

Introduction

It was as early as 1950 that Isamu Yamada published a paper "National Accounting in Norway" in a journal *Economic Review* (*Keizai Kenkyu*) in Japanese.[1] He introduced Norwegian national accounting system to Japanese academics focusing on Ragnar Frisch's ecocirc system by using figures below reproduced here. Yamada also mentioned Norwegian white paper system of national accounts that Odd Aukrust contributed to much and was of Anglo-American or Keynes-Stone type. The present paper is intended to be a tribute to the two great Norwegian figures in national accounting, Ragnar Frisch and Odd Aukrust. It is generally said that national accounting was discovered simultaneously by Frisch and Keynes, while Aukrust served to bridge the two schools of national accounting (Norwegian or Scandinavian and Keynes-Stone) and established a new field of study for national accountants, that is, axiomatic studies of national accounting.

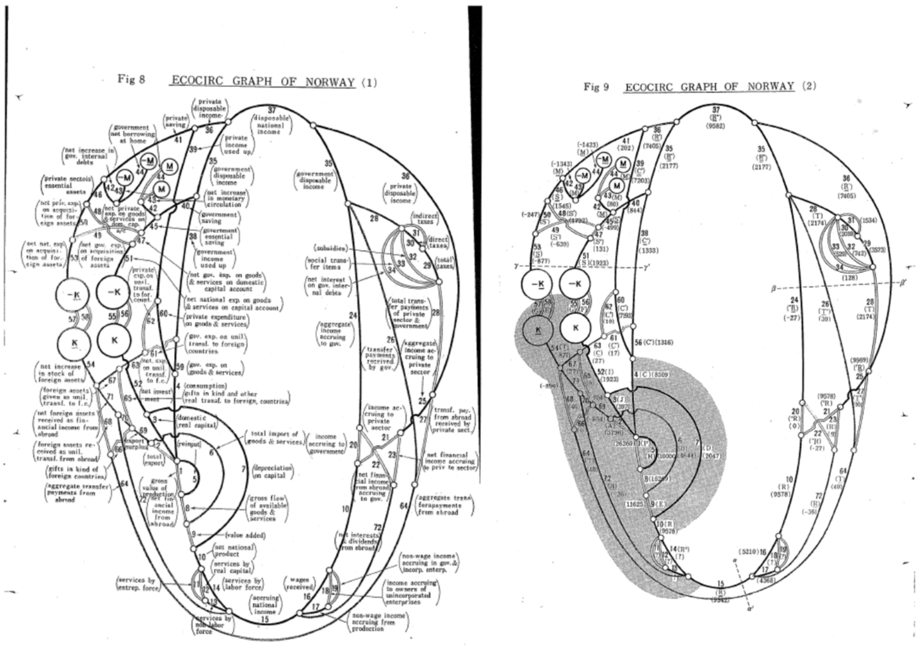

Figure 1: From Yamada [1950]

The present paper is organised as follows. The section following the introduction will deal with the history. In particular, it will be shown that Aukrust's work had a great impact on business accountants including Richard Mattessich as well as national accountants.

[1] Yamada [1950].

The second section discusses Deguchi's axiomatic studies on accounting as one of the examples of the impacts given by Aukrust's work. He focused and axiomatised bookkeeping or business accounting algebraically and formulated the internal models of economic agents to proceed to analyse the economy as an agent-based complex system.

The third section discusses Sakuma's axiomatic studies on national accounting. It was Frisch rather than Aukrust that he relied upon. Seeing figures above reveals that Frisch considered the circular flow of the economy to be a digraph. Sakuma formulated national accounting as an Eulerisation of a digraph.

Concluding remarks are given at the end of the paper. An appendix will be given to a formal presentation of the definitions and axioms for the graph-theoretical axiomatisation to national accounting.

1. A brief history of axiomatic approaches to national and business accounting

The Norwegian school, represented by Frisch and Aukrust, had a great influence not only on the development of national accounting theory, but also on the development of accounting research, including business accounting. As Bjerve [1996, p.4] wrote, Frisch succeeded in defining a system of macroeconomic concepts surprisingly similar to that of modern national accounts. Also, Frisch emphasized the need for standardising the macroeconomic concepts (Bjerve[1996, p.6]). Thus, the contribution made by Frisch and the Oslo school to national accounting and macroeconomics is unexpectedly large. The distinction between real concepts and financial (nominal) concepts was one of those most focused upon.

National accounting, which is also known as social accounting or national income accounting, is a relatively new field of research that was formed and developed during 20th century. In the 1940s, it was recognized as an accounting system (Hicks [1942]); the efforts made by the OEEC and the United Nations to ensure the standardization of national accounting progressed since then. At present, national accounting information is collected and provided to the public in each country. Further, it has attained an important position in the administration of the national economy due to its role in policy decision-making.

The development of national accounting has brought about a new perspective in the field of accounting research, which has been focused on business accounting. As national accounting is "a third 'rediscovery' of accounting, this time by economists" (Littleton, [1958, p. 248]), studies have been analyzing the relationship between national accounting and business accounting since the 1950s. In particular, research aimed at constructing the general theory of accounting opened up a new horizon in accounting academics. This theory focuses on the isomorphism between national and business accounting. Researchers in the field of business accounting such as Richard Mattessich and Yuji Ijiri studied the general theory of accounting, considering both business and national accounting; both of the general theories were greatly influenced by the Aukrust's axiomatic system. Therefore, we explain the axiomatic system proposed by Aukrust, Mattessich, and Ijiri below and present the trend of current accounting research, generated by such studies on the general theory.

1-1. Aukrust's axiomatic system

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The ecocirc graph presented by Frisch [1943] (cf. Figure 1) is the starting point of the axiomatisation of national accounting. Further, as will be discussed in section 3, ecocirc is a digraph presenting circular economic flows. Frisch [1943] was the first contributor in the world to demonstrate that this structure of circular economic flows can be expressed in any algebraic, graphical, and accounting form; this can be regarded as the foundation of the development of national accounting.

Aukrust developed his national accounting theory by incorporating Frisch's [1943] theory. Regarding the axiomatic system, Aukrust [1992] said that "the concepts used in national accounting must be established in an axiomatic manner" [p. 6]; it is considered that Aukrust recognized the importance of adopting an axiomatic method to logically establish definitions, classifications, and measurements for national accounting.

According to Aukrust [1992], when he became the head of national accounting at the Norwegian Central Statistics Office, based on Frisch's ecocirc concept, the Norwegian national accounting system adopted the standardized accounting structure proposed in the United Nation's paper, commodity flow accounting developed in Denmark, and input output table proposed by Leontief; this culminated in Aukrust's doctoral dissertation that is written in Norwegian. The appendix section of this dissertation includes an axiomatic foundation for the theory of national accounting and its translation provided by the Japanese accounting researcher Yoshiaki Koguchi (Aukrust [1955]; (transl.) Koguchi [1998]).

Aukrust's axiomatic system consists of 20 postulates and 80 theorems. The 20 postulates are listed in Table 1, and they are classified into four categories: real circulation, financial circulation, the interplay between the real and financial circulation, and measurement issues. Although Aukrust [1955] employed the set theory and algebra to explain each axiom, Table 1 presents them in the form of sentences, which is in line with Aukrust [1966].

Table 1 Postulates presented in Aukrust [1966]

The real circulation
<Postulates establish categories.> I. *Sector* or transactor; a sub-set of the sectors define the domestic economy. II. *Time*, conceived of as consecutive "time intervals" separated by "points of time. III. *Economic objects*, of which there are two kinds (real or financial). <A postulate establishes a relationship between sectors, time, and real objects.> IV. For every real object, at any one point of time during its existence, one_and only one_sector can be denoted as the "owner" of the object. <Postulates describe the transactions which real objects may be subject to, and which are of interest in national accounting> V. *Production*, conceived of as transformation processes whereby real objects are "created" (star to exist) at the time as other real objects, used as inputs, are cancelled out (cease to exist). VI. *Final Consumption*, conceived of as processes whereby real objects are cancelled out (cease to exist), other than by being used as inputs. VII. *Change of ownership.* <Postulates finally induced by the postulates V, VI and VII.> VIII. No real object comes into existence by any other way than by being produced. IX. No real object can be involved in more than one transaction of the types described by postulates V through VII in the shortest time interval we have under consideration.
The financial circulation (in addition to the 9 postulates listed above.)
<A postulate establishes a relationship between sectors, time, and financial objects.> X. For every financial object, at any point of time during its existence, one creditor sector and one debtor sector can be identified. <Postulates describe the transactions in which financial objects can be involved and which are of interest in national accounting.> XI. Financial objects can be *created*, or XII. *Cancelled out* (cease to exist), or XIII. Undergo a *change of creditor*, or XIV. Undergo a *change of debtor.* <A postulate finally induced by the postulates XII, XIII and XIV.> XV. No financial object can be involved in more than one transaction of the types described by postulates XI through XIV in the shortest time interval we have under consideration.
The interplay between the real and the financial circulation (in addition to the 15 postulates listed above.)
<Postulates which introduce the distinction between requited and unrequited transactions> XVI. A real flow from one sector to another is always associated with a financial contribution in the opposite direction. XVII. A financial contribution from one sector to another *can* (but need not) be associated with financial contributions in the opposite direction. XVIII. No financial contribution from one sector to another can be associated with both a real flow and a financial contribution in the opposite direction.

The problem of measurement. Definition of value concepts.
XIX. There exists a "*national accounts price list*" in the sense that for every object, real or financial, one non-negative, rational number is given which can be taken to express the value of the object.
XX. Two requited flows (real/financial or financial/financial) always have the same value (the postulate of "the preservation of values in exchange").

Source: Based on Aukrust [1966, pp. 181-187].

Koguchi [2017] summarized Aukrust's adaptation of Frisch's [1943] ecocirc theory in establishing his axiomatic system by (1) adopting the methodology of theorizing national accounting as an axiomatic system, (2) using the definitions and basic concepts employed in the ecocirc theory, (3) dividing the economic circulation into real and financial circulation, (4) setting the object of measurement as an economic object consisting of real and financial objects, and (5) distinguishing a transaction based on whether it has been paid or not (requited and unrequited transaction). Thus, it can be understood that Frisch's thought was inherited by Aukrust through the systematization of the national accounting theory using the axiomatic system.

1-2. Mattessich's axiomatic system

Mattessich [1956] argued for the need of a deductive theory that embraces both business accounting and economics, stating that "(b)oth branches examine the individual economic cell as well as the entire economic body of a country" [p. 551]. It was just one year after Aukrust [1955] published his doctoral dissertation in Norwegian. On the basis of the fact that economics is theoretically constructed in an axiomatic framework, Mattessich proposed a plan of axiomatic approach by using matrix algebra for the first time in 1957 for constructing the general theory of accounting. Consequently, he tried to refine the axiomatic system by using the set theory; the results were compiled in Mattessich [1964].[2]

He positioned accounting as a subdiscipline of economic sciences, which is considered as the superdiscipline, and then defined accounting as "a discipline concerned with the quantitative description and projection of the income circulation and of wealth aggregates by means of a method based on the following set of basic assumptions" (Mattessich [1964, p. 19]). The "duality principle," which is positioned as a basic principle, "asserts that there exist economic events which are isomorphic to a two-dimensional

[2] Following Mattessich [1964], Mattessich had been working on the refinement of axiomatic systems. However, this paper is written based on Mattessich [1964] because the basic idea of the system is shown in this work.

classification of a value within one set of classes" (Mattessich [1964, p. 27]). Mattessich proposed this principle following the references of national accounting systems, such as national income accounts, input output tables, and the flow of funds accounts. This principle is considered as a broad concept along with the double entry principle in double entry bookkeeping.

Eighteen basic assumptions are presented based on the duality principle in Table 2; it lists each basic assumption and their explanations.

Table 2 Basic assumptions presented in Mattessich [1964]

1.	Monetary Values.	There exists *a set of additive values*, expressed in monetary units; this set is isomorphic to the system of integers plus the number zero.
2.	Time Intervals.	There exists *a set of elementary, additive time intervals*.
3.	Structure.	There exists a structured set of classes reflecting significant categories of an entity.
4.	Duality.	For all *accounting transactions* it is true that a value is assigned to a three-dimensional concept consisting of *two* accounts and a time instance.
5.	Aggregation.	Every balance assigns a value to an ordered pair; the latter consists of pertinent account and the above stated period which starts with the accounting period.
6.	Economic Objects.	There exists *a set of economic objects*, whose value and physical properties are subject to change.
7.	Inequity of Monetary Claims.	There exists a custom to enter debts with the understanding to *redeem them in legal tender at face value*.
8.	Economic Agents.	There exists a set of economic agents who set specific goals to an accounting system, command resources, and make plans and decisions with regard to economic actions.
9.	Entities.	There exists *a set of entities* setting the frame for economic actions.
10.	Economic Transactions.	There exists *a set of empirical phenomena*, called *economic transactions*. Each of these transactions assigns, by means of empirical hypothesis, a value to an ordered pair of transactors and a time instance.
11.	Valuation.	There exists a set of hypotheses determining the value assigned to an accounting transaction.
12.	Realization.	There exists a set of hypotheses, specifying which of the following three mutually exclusive effects are exercised by *a change of an entyty's economic objects(s)*.
13.	Classification.	There exists a set of hypotheses required to establish a chart of accounts.
14.	Data Input.	There exists a set of hypotheses required to determine the form of data input and the level of aggregation for which accounting transactions are to be formulated.
15.	Duration.	There exists a set of hypotheses about the expected life of the entity under consideration, and the duration of individual accounting periods or subperiods.
16.	Extension.	There exists a set of hypotheses specifying the empirical conditions under which two or more accounting systems can be consolidated and extended
17.	Materiality.	

	to a more comprehensive picture.
18. Allocation.	There exists a set of hypotheses determining if and when an economic transaction or related event is *to be reflected by an accounting transaction*. There exists a set of hypotheses determining the *allocation* of an entity's economic objects or flows of services *to subentities* and similar categories.

Source: Summarized by the authors based on Mattessich [1964, pp. 32-45.]

Further, in order to clarify the concept of these basic assumptions, he attempted to axiomatize accounting by applying set-algebraic formulation based on two key concepts: ownership and debt-claim. Here, we do not mention the details of set-algebra, although it is useful to point out how he used Aukrust's axiomatic system in axiomatization (cf. Nose [1971]).

Basic assumption 6 follows from postulate III (in Table 1) of Aukrust's system, as it is divided into real and financial objects. Following basic assumption 10, transactions are divided into "requited" and "unrequited" as propositions; hence, it can be said that it is an application from postulate XVI of Aukrust's system. Basic assumption 9 specifies the extent of the entity and can be considered as a refined treatment of the "sector" in Aukurst's preposition I, IV, X, XVI, XVII, and XVIII. Further, assumptions 2 and 15 related to the refining of the axiom of "time" and were lined to the definition of "accounting period."

Thus, it can be considered that Mattessich's axiomatic system is strongly influenced by Aukrust's axiomatic system.

1-3. Ijiri's axioms of national accounting

Yuji Ijiri is a Japanese robust historical cost accounting theorist who contributed to accounting research in America. He developed business accounting theory, which was characterized by historical cost accounting theory, using axiomatic systems. This study was inspired from the general theoretical study of Aukrust and Mattessich using axiomatic systems, and Ijiri's axiomatic system that features the use of three axioms: control, quantities, and exchanges. Further, Ijiri's [1979] study clarified the general theory that bridges the gap between national and business accounting using the above three axioms, based on his "belief that corporate accounting and national accounting must derive from the same foundation" (Ijiri [1979, p. 223]).

Ijiri [1979] considered both national and business accounting to display the characteristic of multisector accounting, which consists of every economic sector and business division. This study considers the multisector accounting structure as the general structure of accounting and examines its application to national accounting. The axiomatic system of multisector accounting is explained below. First, the three basic and fundamental axioms are explained in Table 3.

Table 3 Basic axioms presented in Ijiri [1979]

1. Axiom of Control.	For any sector, the set of all resources under its control at a given point of time can be uniquely identified at that point or later. In addition, no two sectors can control a given resource simultaneously.
2. Axiom of Quantities.	All resources under the control of any sector presently or in the past can be uniquely partitioned into a set of classes. Associated with each class is a unique quantity measure, satisfying the additivity and indifference conditions.
3. Axiom of exchanges.	Changes in resources under any sector's control can be uniquely partitioned into a set of exchanges, each of which identifies the resources exchanged and the sectors involved.

Source: Ijiri [1979, p. 213].

Ijiri stressed the importance of quantity as well as the financial amount of goods exchanged. Based on this view, a transaction is journalized as follows (the counterparty record where debit and credit are calculated as the opposite entries):

(Dr.) Sector 1 - Wheat 1 million bushels

(Cr.) Sector 2 - Cash $4 million

Consequently, three basic concepts of economic sectors, economic resources, and economic activities are introduced as fundamental elements of multisector accounting. These correspond to the axiom of control, quantity, and exchange. The matrix B^t which considers the resources under the control of each sector is determined by the control and quantity axioms. The rows of this matrix represent the sectors, while the columns represent the resources considered. Further, the classes of n resources [goods] controlled by m sectors are arranged in quantity units. This matrix is expressed as follows:

$$B^t = \begin{bmatrix} b_{11}^t & b_{12}^t & \cdots & b_{1n}^t \\ b_{21}^t & b_{22}^t & \cdots & b_{2n}^t \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1}^t & b_{m2}^t & \cdots & b_{mn}^t \end{bmatrix}$$

Each element of the Matrix B^t changes with time, but it is determined through the axiom of exchange. That is, as shown by the following equation, the variation in the elements of the matrix B^t caused by the exchange is expressed as the activity matrix A.

$$B^t - B^{t-1} \equiv A^t = A_1^t + A_1^t + \cdots + A_l^t$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The activity matrix A is also arranged with the sectors presented in rows and classes of resources presented in columns. Thereafter, the following five types of economic activities recorded in the activity matrix A are presented.

1. Trading: $\begin{bmatrix} +q_1 & -q_2 \\ -q_1 & +q_2 \end{bmatrix}$ 2. Production: $[+q_1 \quad -q_2]$ 3. Transfers: $\begin{bmatrix} +q_1 \\ -q_1 \end{bmatrix}$

4. Consumption: $[-q_1]$ 5. Generation: $[+q_1]$

For example, trading transaction, considering the transaction of the above journal entry, is expressed as follows:

$$\begin{bmatrix} +q_1 & -q_2 \\ -q_1 & +q_2 \end{bmatrix} = \begin{bmatrix} wheat\ increase & cash\ decrease \\ wheat\ decrease & cash\ increase \end{bmatrix}$$

The above transaction is described in the activity matrix A as follows:

$$A = \begin{bmatrix} +q_1 & -q_2 & \cdots & 0 \\ -q_1 & +q_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Every time a transaction occurs, the records of economic activities are divided into an activity matrix A_k^t according to the exchange axiom, such that the activity matrix A of the national economy shown in Table 4 can be completed.

Table 4 Activity matrix presented in Ijiri [1979]

	Production	Consumption	Accumulation	Rest of the World
1. Purchases of factor services 2.Production				

… 7.Consumption 8.Depreciation				
9.Saving - investment				

Source: Based on Ijiri [1979, p. 220].

Ijiri's [1979] axiomatic system is based on the quantitative measurement of physical flows, which is an important aspect of his model. He points out that traditional national accounting only focuses on financial flows and excludes the flows of physical resources. Ijiri acknowledges Aukrust's axiomatic system for also considering material or physical flows; however, he believes that the "postulate of the preservation of values in exchange" in axiom XX [in Table 1] presented by Aukrust is unnecessary when considering the formulation of his axiom because it uses a quantitative axiom. In addition, regarding the postulate of economic object, he states that real objects are more important than financial ones, and that the exchange or transfer between physical objects are not considered.

These criticisms provided the theory on which they are based differs from the movement of axiomatization from the viewpoint of national accounting such as that of Aukrust and the movement of axiomatization from the viewpoint of business accounting such as that of Ijiri in that the former is based on economics and the latter is based on historical cost. However, as Koguchi [1985] stated from the viewpoint of national accounting, although physical information is important, if too much emphasis is placed on it, the economic circulation system cannot be completed by it. Hence, from the perspective of national accounting, Aukrust's thought of placing importance on monetary information in the axiomatic system must be adopted.

1-4. Japan's case

Mattessich examined national accounting from the viewpoint of business accounting and aimed to establish the general accounting theory that includes both national and business accounting. However, it is said that the research on the general accounting theory by business accounting scholars was almost ignored by both economists and business accounting scholars at that time (Powelson [1955]; Mattessich [1964]). In Japan, however, it was strongly inspired by the considerable works of Mattessich and Ijiri, following which many Japanese accounting scholars have analysed and examined their works and have been trying to conduct research that includes both business accounting and national accounting. It is assumed that there is a logical reason behind why such research was developed in Japan. Kiyoshi Kurosawa, one of the most famous Japanese accounting scholars of the 20th century, published his first research paper analyzing macroeconomics from the perspective of accounting in 1932([Kurosawa, 1932]). However, internationally, it is not well-known that a scholar at that time envisioned accounting for the national economy. Since then, Kurosawa's continued publication of a theory that encompasses accounting for economics produced a succession of accounting scholars in Japan who applied the theory in the fields of business and national accounting, which is thought to have led to the continuation of research aimed at forming a general theory of accounting. For example, Nose [1961] had developed the theory of isomorphism of national and business accounting, while Koshimura [1968] proposed a new method of bookkeeping by using matrix algebra that incorporated the input output table model into business accounting.

By summarizing the above discussion, it can be concluded that studies by two Norwegian scholars have led to the concept of accounting that includes both micro and macro accounting in line with the formulation of a general accounting theory, following Mattessich and Ijiri. More recently, a new accounting area, known as meso accounting, has been developed; it is positioned in the middle from both micro and macro viewpoints. Meso accounting is developed in the form of measuring and reporting the situation of resources, stocks, and flows in areas such as water resource accounting, carbon accounting, and biomass accounting.

Further, Japanese national accounting scholars Sakuma and Deguchi also proposed the formation of a general theory using graph-theoretical and algebraic approaches. In the following, the modern development of the algebraic and graph-theoretical approaches will be explained.

2. An Algebraic approach to axiomatisation of business and national accounting

2-1. Accounting Postulate and the Axiomatisation of Bookkeeping System

When we discuss the axiomatization of the bookkeeping system, two concepts are often confused: the concept called postulate and axiom. There is no significant dissociation between them as long as they are used for the foundations of geometry. In geometry, assumptions without proof were divided into postulates and axioms, with the former used for more geometric assumptions. Still, in the context of modern mathematical proof theory, there is no difference between two concepts that are unified into the concept of axiom.

On the other hand, postulates in accounting mean different concepts. Accounting principles are standards that are used as norms to govern how transactions are processed and recorded.

Aukrust discussed an axiomatic approach to National Accounting [Aukrust, 1966]. In his approach, he has introduced twenty postulates to derive a national accounting system with algebraic relations between values of the System of National Accounting. His usage of postulates is similar to the accounting principle.

This chapter introduces the concept of double-entry state space and its change as an extended stock-flow dynamics ([Deguchi, 1986; 2002]). Then we characterize the double-entry state-space structure as an algebraic point of view. As a kind of algebraic formulation, Matrix Accounting by Richard Mattessich is very famous. He has clarified how matrix form can represent the tabular form of bookkeeping system and its calculation on table. He also describes many types of axiomatisation in his book entitled " Two Hundred Years of Accounting Research " that cite my algebraic axioms ([Mattessich,2007]). He did not distinguish algebraic characterization from a characterization of accounting postulate.

There exist several types of Matrix or Vector-based characterization of the bookkeeping system. These approaches use a mathematical framework of state description that already exists and try to show how the calculations of bookkeeping can be represented by the framework, where the algebra is already characterized by its proper axioms such as the axiom of linear algebra.

In contrast, we characterize the bookkeeping system itself by its proper axioms. Where we can characterize the concept of "Minus Stock" such as Liabilities and of "Double Entry State Space" by the axioms, then, for example, we can explain why there exist two mutually exclusive sets of accounting titles called "Debit Side" and "Credit Side" from the axioms as a proposition.

2-2. Exchange Algebra and its Axiomatic Characterisation

We show the concise definition of Exchange Algebra. The detail is shown in the book entitled "Economics as An Agent-based Complex System" [Deguchi, 2004].

[Definition 2.1] A Set of Accounting Bases

Let $\Lambda = \{e_1, e_2, \dots e_n\}$ be a set of suitable title of accounts.

$[\Lambda]$ denotes the commutative free semigroup with a unit element 0, generated by a summation of elements of Λ.

Then Λ is called a set of bases, and e_i is called a base of account.
What we expressed as bases is an abstraction of an account such as <cash, Yen> or <Apple, Kg>.

[Definition 2.2] Dual Base and Expansion of Base
Let $\Lambda = \{e_1, e_2, \dots e_n\}$ be a set of bases, which consist of title of accounts.
Then new base $\hat{}e_i$ is introduced from $e_i \in \Lambda$, which is called a dual base of e_i.
$\hat{}\Lambda = \{\hat{}e_i \mid e_i \in \Lambda\}$ is called a set of dual bases.
$\Gamma = \Lambda \cup \hat{}\Lambda$ is called a set of extended bases of Λ.

[Definition 2.3] Set of Accounting Bases
Let $\Lambda = \{e_1, e_2, \dots e_n\}$ be a set of bases.
Let $\Omega = \{$<assets>, <liability>, <equity>, <revenue>, <expense>$\}$.
If there exist onto function $k: \Lambda \dashrightarrow \Omega$, which classify elements of Λ, then$\Lambda$is called a set of accounting bases.
The definition shows that an account list can be classified by the elements of Ω.

[Definition 2.4] Accounting Vector Space and Accounting Algebra
Let Λ be a set of accounting bases.
Let $\Gamma = \Lambda \cup \hat{}\Lambda$ be a set of extended bases of Λ.
Then $[\Gamma]$ is called an accounting vector space, which consists of accounting bases Λ. An element of $[\Gamma]$ is called an accounting vector.
An accounting vector space is not a linear space. However, many of its properties are similar to linear space. A similar concept of norm on linear space can be introduced into accounting vector space.
If Γ satisfies the following Axiom 2.1, then $[\Gamma]$ is called an exchange algebra. In the following section, we assume that an accounting vector space satisfies the axiom of exchange.

[Axiom.2.1] Axiom of Exchange
(1) $\forall a,b \quad a \Leftrightarrow b \equiv \hat{}a \Leftrightarrow \hat{}b$
(2) $\forall a,b,c \quad a \Leftrightarrow b$ and $b \Leftrightarrow c \rightarrow \neg(a \Leftrightarrow c)$
(3) $\forall a,b \quad a \Leftrightarrow b \equiv b \Leftrightarrow a$
(4) $\forall a,b \quad a \Leftrightarrow b \rightarrow \neg(a \Leftrightarrow \hat{}b)$
(5) $\forall a,b,c \quad \neg(a \Leftrightarrow b)$ and $\neg(b \Leftrightarrow c) \rightarrow \neg(a \Leftrightarrow c)$

(6) $\forall a \exists b \quad a \Leftrightarrow b$

Where a, b, c $\in \Gamma$, and ¬ means logical symbols of negation.

Let a =<cash, Yen>, b = ^<apple, Yen>, and c = <account receivable, Yen>'

Axiom (1) means that if the transaction "an apple is sold to acquire cash" is significant and acceptable (^<apple, Yen>⇔<cash, Yen>) then the transaction "cash is paid to purchase an apple" is also significant and acceptable (^<cash, Yen >⇔<apple, Yen >).

Axiom (2) means that if the transaction "an apple is sold to acquire cash" satisfies the exchange relation (^<apple, Yen >⇔<cash, Yen >) and the transaction "an apple is sold and receive accounts receivable" also satisfy the relation (^<apple, Yen >⇔<accounts receivable, Yen >) then the transaction to acquire cash and accounts receivable does not satisfy the relation (¬<cash, Yen >⇔<accounts receivable, Yen >).

Axiom (3) means reflective law.

Axiom (4) insists that if the transaction "an apple is sold to acquire cash" satisfies the exchange relation (^<apple, Yen >⇔<cash, Yen >) then the transaction to acquire cash and apple at the same time does not satisfy the relation (¬<apple, Yen >⇔<cash, Yen >). However, we do not insist that it is impossible to describe the transaction "the cash and apple are obtained at the same time as a free gift" by exchange relation. The economic activity of "the cash and apple are obtained at the same time as a free gift" is described as the transaction of the occurrence of profit by double-entry bookkeeping.

It is described as <cash, Yen > ⇔ <profit, Yen> and <apple, Yen > ⇔ <profit, Yen> in our frame work.

Axiom (5) insists that the no exchange relation (¬⇔) satisfies transitive law.

For example, if there is no exchange relation between <cash, Yen > and <apple, Yen > (¬<cash, Yen > ⇔ <apple, Yen >) and there is no exchange relation between <apple, Yen > and <accounts receivable, Yen > (¬<apple, Yen > ⇔ <accounts receivable, Yen >), then the axiom insists that there is no exchange relation between <cash, Yen > and <accounts receivable, Yen > (¬<cash, Yen > ⇔ <accounts receivable, Yen >).

Axiom (6) insists that for any account name "x" there exists a certain account name "y" in Γ, and "x" and "y" satisfy the exchange relation. As a result, these six axioms characterize the significant and acceptable transaction.

[Proposition 2.1]

Let Λ'= {PS, IN, MS, OUT}.

Let g: Λ→Λ' be an onto function.

Let ⇔ be a relation on Λ, which satisfies the following conditions (1) to (5).

Then the relation ⇔ can be extended to the exchange relation on the extended basis Γ uniquely.

(1) If g(x) = PS and g(y) = IN then x ⇔ y and y ⇔ x

(2) If g(x) = PS and g(y) = MS then x ⇔y and y ⇔ x

(3) If g(x) =OUT and g(y) = IN then x ⇔ y and y ⇔ x

(a) If g(x) = OUT and g(y) = MS then x ⇔ y and y ⇔ x

(5) Otherwise ¬x ⇔ y

Notice: <assets>, <liability>, <equity>, <revenue> and <expense> are interpreted as PS(Plus Stock), MS(Minus Stock), MS(Minus Stock), IN(In flow) and OUT(Out flow) respectively.

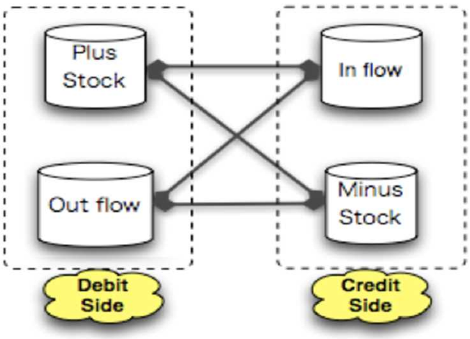

Figure 2: Exchange Relation on {PS, IN, MS, OUT}

[Example 2.1] Tabular and Algebraic Expression

Debit Side	Credit Side	Debit Side	Credit Side
50 Cash	50 Liabilities	50 Liabilities	50 Cash

x=50<Cash, Yen>+ 50<Liabilities, Yen> y=50^<Cash, Yen>+ 50^<Liabilities, Yen>

[Example 2.2] Algebraic Calculation

X1=4000<Orange, Yen> + 4000^< Cash, Yen>

X2=8000<Apple, Yen> + 8000^< Cash, Yen>

X3=2000< Utility bill, Yen> + 2000^< Cash, Yen>

X4=5000<Rent, Yen> + 5000^< Cash, Yen>

X5=8000<Wages, Yen> + 8000^< Cash, Yen>

X6=1000^<Orange, Yen> + 2000< Cash, Yen> + 1000<Profit, Yen>

X7=3000^<Orange, Yen> + 8000< Cash, Yen> + 5000<Profit, Yen>

X8=8000^<Apple, Yen> + 30000< Cash, Yen> + 22000<Profit, Yen>

Then we can calculate total balance of balance as

~{ x1+x2+x3+x4+x5+x6+x7+x8 } = 2000< Utility bill, Yen> + 5000<Rent, Yen> + 8000<Wages, Yen> + 13000< Cash, Yen> + 28000<Profit, Yen>

2-3. Multi-Dimensional Extension of Exchange Algebra and Its Application

We introduce a bookkeeping system with material flow information, that is called a multi-dimensional bookkeeping system on exchange algebra, by extending exchange algebra with accounting units for materials. We also introduce the multicurrency description of the bookkeeping system on exchange algebra.

[Definition 3.1] Multi-Dimensional Description and Price Function
We introduce units for materials as an example. We use "Kg", " kWh", "Minutes", "Hours", "pieces" for measuring materials, services, energy, human capital, investment, and related accounting titles.

[Example 3.1] Algebraic Expression of Multi-Dimensional Bookkeeping System

Debit Side	Credit Side	Debit Side	Credit Side
Apple 500 Yen	Cash 500 Yen	Apple 2 Kg	Cash 500 Yen

X1=500<Apple, Yen> + 500^< Cash, Yen>   x2=2<Apple, Kg> + 500^< Cash, Yen>

[Example 3.2] Algebraic Multi-Dimensional Expression of Production Process
The production process consists of the inputs such as materials, human capital service, and capital equipment service and the output such as product and by-product. We can describe the production process as an input-output process by the algebraic multi-dimensional expression as follows.
We focus on Cooper Sheet Cutting Task by NC Milling Machine. The inputs of the process consist of cooper sheet (materials), human capital service, and capital equipment service. Human capital service is an output of human capital production process where the input of the process is labor costs that are treated as an asset. Capital equipment service is an output of capital equipment service production process where the input of the process is energy and depreciation of NC Milling Machine.

For example, the production process is shown by algebraic multi-dimensional accounting as follows.
X1=1< Cutted Cooper Sheet, Pieces> + 2<Cooper By-product, Kg> + 8^< Cooper Sheet, Kg> + 1^<Cutting Capital Equipment Service, Hours> + 0.2^<Operation Human Capital

Service, Hours>

X2=1<Cutting Capital Equipment Service, Hours> + 1<Depreciation Reserve, Hours> + 2.5^<Electrical Energy, kWh >

X3=1<Human capital service, Hours> + 1^<Labor Costs, Hours>

Monetary-based accounting can be reduced from algebraic multi-dimensional accounting by giving price information as follows. Let assume that the price of Cooper Sheet =1000 Yen/Kg, Cooper By-product=400 Yen/Kg, NC Machine Depreciation = 1000 Yen/Hour, Electrical Energy = 2000 Yen/kWh, Labor Costs(Wage) = 2000 Yen/Hour, and then we can convert multi-dimensional algebraic expression to monetary expression as follows.

Y1=a<Cutted Cooper Sheet, Yen> + 800<Cooper By-product, Yen> + 8000^< Cooper Sheet, Yen> + 6000^<Cutting Capital Equipment Service, Yen> + 400^<Operation Human Capital Service, Yen>

Y2=6000<Cutting Capital Equipment Service, Yen> + 1000<Depreciation Reserve, Yen> + 5000^<Electrical Energy, Yen>

Y3=400<Human capital service, Yen> + 400^<Labor Costs, Yen >

We can also get the cost of the Cutted Cooper Sheet as follows.

Projection[Debit Side](Y1)=a+800

Projection[Credit Side](Y1)=8000+6000+400

Then a=8000+6000+400-800=13600

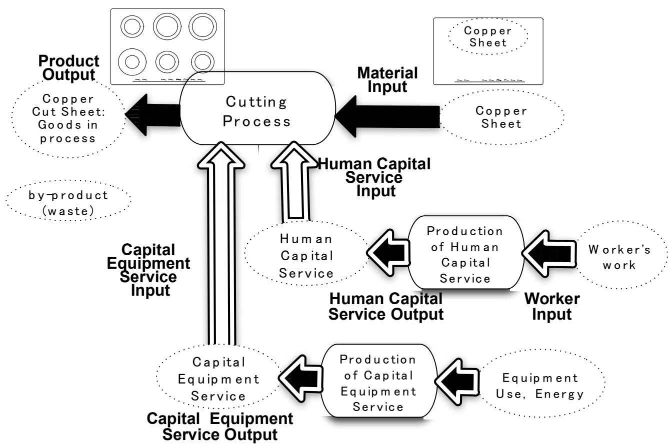

Figure 3: Production Process described by Multi-dimensional Accounting

[Example 3.2] Multi-Dimensional Algebraic Expression of Energy Service Accounting

In the IoT era, double-entry description is to be widely used for service process described as follows. First, the utility bill is replaced by an energy asset. Second, energy asset is used to produce energy service such as air conditioning service. Third, the energy service is used as an input of certain capital equipment services or is used as consumption.

(1) Energy Use Description as a Utility bill

Debit Side Credit Side

Utility bill (Electrical Energy) 20 Yen Cash 20 Yen

Cost 20 Yen Utility bill (Electrical Energy) 20 Yen

Y1=20< Utility bill, Yen>+20^<Cash Yen>

Y2=20^< Utility bill, Yen>+20<Cost Yen>

(2) Energy Use Description as production of energy service by energy asset

Debit Side Credit Side

Electrical Energy as Asset 20 Yen Cash 20 Yen

Air conditioning Service 20 Yen Cash 20 Yen

AC Consumption 20 Yen Air conditioning Service 20

Cost 20 Yen AC Consumption 20 Yen

Y2=20< Electrical Energy, Yen>+^20<Cash Yen>

Y3=20^< Electrical Energy, Yen>+20< Air conditioning Service, Yen>

Y4=20<AC Consumption, Yen>+ 20^< Air conditioning Service, Yen>

Y5=20^<AC Consumption, Yen>+ 20< Cost, Yen>

(3) Electrical Energy Use Description as production of energy service by energy asset and its Multi-Dimensional Algebraic Expression

Debit Side Credit Side

Electrical Energy as Asset 1 kWh Cash 20 Yen

Air conditioning Service 1-hour Electrical Energy as Asset 1 kWh

AC Consumption 1 hour Air conditioning Service 1 hour

Cost 20 Yen AC Consumption 1 hour

Y6=1<Electrical Energy, kWh >+^20<Cash Yen>

Y7=1^< Electrical Energy, kWh >+1< Air conditioning Service, hour>

Y8=1<AC Consumption, hour>+ 1^< Air conditioning Service, hour>

Y9=1^<AC Consumption, hour>+ 20< Cost, Yen>

Using "Multi-Dimensional Algebraic Expression," we can know how electrical energy is used for what services. If we can aggregate all electrical energy services of a country in each hour, then we can construct national energy accounting where we can know what types of energy services are consumed by electrical energy at the country in each hour.

2-4. Conclusion

In this chapter, we have introduced exchange algebra and its axiomatic characterization. In the detail of the algebra and its application to National Accounting, please refer to [Deguchi, 2004].

By using exchange algebra, we have characterized double-entry state-space, which is the natural and unique extension of stock-flow dynamics under minus stock and double-entry extension. Furthermore, the multi-dimensional expression of exchange algebra makes it possible to characterize many types of service production, transaction, consumption, and investment by its natural measurement units and monetary units.

In IoT Ara, it will become possible to capture many types of Point of Event data as real-time data that include production, transaction, consumption, and investment.

Recently the concept of a real-time economy is becoming popular.

Exchange algebra can give a suitable architecture for the next transaction base real-time administrative and organizational management infrastructure by providing algebraic multi-dimensional state-space description. The algebraic multi-dimensional state space can treat not only monetary-based transactions but also multi-dimensional transactions. So far, we have not been able to use double-entry state space for energy, human capital, waste, CO2, and many services and materials for its management. Because the description is regarded as a physical description. Then the description is

We have already proposed the concept of real-time statistical surveillance depending on transaction data as a handout of the fourth working group meeting (11th) of the National Statistics Committee[Deguchi, 2008]. Many types of "Point of Event Data," such as point of sales event data that is called POS data, can be used to construct national statistics surveillance. Now we can use e-invoice data to capture point transaction events. Then we can survey commodity flow data instead of a government survey of Commodity Flow Survey(CFS). E-invoice can be expressed as a serialized form of an invoice expressed by using multi-dimensional exchange algebra. We can deserialize the serialized e-invoice. Then the data can be recognized as a calculable instance object of exchange algebra. As a result, we can construct real-time transaction data from an algebraic expression of e-invoice for economic policymaking. We call it real-time economy.

3. A graph-theoretical approach to axiomatisation of national accounting

As we have seen in previous sections, Ragnar Frisch devised a digraph (directed graph) that he named “ecocirc,” by which he gave the images of the circular flow of the economy. It is just this Norwegian contribution made by Frisch that Sakuma [2006a, 2006b] crucially relied on to seek his approach to the axiomatisation of national accounting.

He defined "the Economy" as a digraph following Frisch and "national accounting system" as an "Eulerisation" of the digraph, the Economy. This means that some arcs are added to the original digraph to form an Eulerian trail. The easiest way is cycle completion. For any arc in the digraph, it is easy to see that you can add an arc with opposite direction to form a small cycle, for example.

Let us consider the case of the exchange of equivalents. The economic agent A sells something to the economic agent B. And B pays A for it. This situation will be described in a digraph such as the one shown below, where the right arrow shows A sold something, say, worth $100 to B and B paid $100 in cash.

Figure 4: The exchange of equivalents

You can find that there is a small cycle here. It is well known that Eulerian digraph can be split into cycles that do not have any arc in common. Actually, it is a necessary and sufficient condition for a given digraph to be Eulerian. Another necessary and sufficient condition for a given digraph to be Eulerian is that at each vertex, the indegree equals the outdegree. The latter condition invites us to consider the similarity of accounting systems with Eulerian digraphs. In fact, you can construct an account for each vertex if you put its indegree on the debit side of the account and its outdegree on the credit side just by assuming that each arc has the same value (say, unit value) and multiple arcs are permitted. Any accounting system can be presented by using an Eulerian digraph and any Eulerian digraph can be deemed to be an accounting system.

However, the case of the exchange of equivalents is not always relevant. For example, the government imposes income taxes. People pay her the money. In the SNA, in such cases, the rule is to put an arc (or arcs) called transfers in the opposite direction. It is an example of cycle completion. Placing transfers is national accountants' way of doing it. Business accountants have their own way of cycle completion.

Vertices (in the digraph) correspond to accounts after the digraph is "Eulerised." However, at the first stage before aggregation, vertices are not so like economic accounts as we know them. Each economic agent has at the maximum twice as many vertices as the number of economic categories available in the society.[3] Twice, because there are two kinds of vertices, the vertices of appearance/disappearance (hereinafter referred to as

[3] Here, categories may be real items like various goods and services, financial items like debts and equities or nonfinancial nonreal items including patent rights, trademarks, etc.

A/DA) and those of non-appearance/non-disappearance (hereinafter referred to as N/N).[4] The former type of vertices shows how each item appears or disappears. Arcs coming out of the vertex show the relevant item appears (production or the issuance of debt) and arcs coming into the vertex show the relevant item disappears (intermediate or final consumption or loan repayment).[5]

The latter type of accounts is somewhat similar to places like warehouses at which nothing new appears and nothing disappears. Arcs coming out of the vertex show that the relevant item leaves the economic agent' control and arcs coming into the vertex show the relevant item enters the economic agent' control. For this type of vertices, you can install a meter, so to speak, that measures the amount of stock inside. Given the initial stock, the in-degree of the vertex is added to and the out-degree of it is subtracted from the figure in the meter. Note that each arc has the same value by assumption.

The first aggregation stage is that within an economic agent. For example, if all vertices of appearance and disappearance for goods and services aggregated, we have a new digraph. Here, the aggregation procedure is such that the vertices to be aggregated are deleted and a new vertex appears instead, and the arcs incident to the vertices both of which are included in the set of vertices to be aggregated are omitted from the arc set of the digraph and the remaining arcs that once joined the deleted vertices and vertices outside the set now become incident to the new vertex. By this aggregation stage, the production/consumption part of the digraph will become more familiar one. However, separating production processes from the whole production/consumption processes may not be so easy because it requires linking between inputs and outputs.

Some postulates need to be made for the reorganisation of the production/consumption vertices. However, it is not done in this axiomatisation attempt. In any case, the construction of production accounts seems to be an analytical effort made by national accountants. [6]

At this stage, the Eulerisation steps should be conducted. They are two balancing

[4] In the case of services, there is no N/N type of vertices because services are not able to be stored, while natural items including land, forest, etc. are also real but there is no A/D type of vertices.

[5] It may be interesting to note that this distinction corresponds to the distinction between real accounts and nominal accounts in business accounting.

[6] If it is possible to identify production processes in this economy and some additional conditions are met, it is possible to identify which vertex is "capital." The required conditions could be that goods just produced (arcs on which goods are produced) are headed toward producers' vertices of N/N type. That is, they are at first stacked up in the producers' products inventory. This requirement is imposed for goods being traded as well. For example, they are put on the shops' stockpiles. Then, you can define that a vertex is "capital." if it is a N/N type of vertex that is adjacent to production vertex.

steps. The first step is placing transfer arcs. If you find any imbalance between the number of arcs departing from vertices of one economic agent towards vertices of another, you should place transfer arcs so that they are matched. These transfer arcs should be placed incident to a newly introduced vertex named "transfer." [7]The second balancing step is placing "balancing items" by using a national accounting term. It is necessary to give some explanations on the concept of "the sequence of vertices (accounts)."

To simplify, consider five vertex (account) system. That is, we assume that there are an aggregated real A/DA vertex, an aggregated real N/N vertex, a transfer vertex, an aggregated financial A/DA vertex, and an aggregated financial N/N vertex.

In the figure below, arcs shown are balancing items. For example, the transfer account has net transfer received as its balancing item. As clearly seen from the figure, the sequence of accounts may be considered to be (directed) tree by using a graph theoretical term.

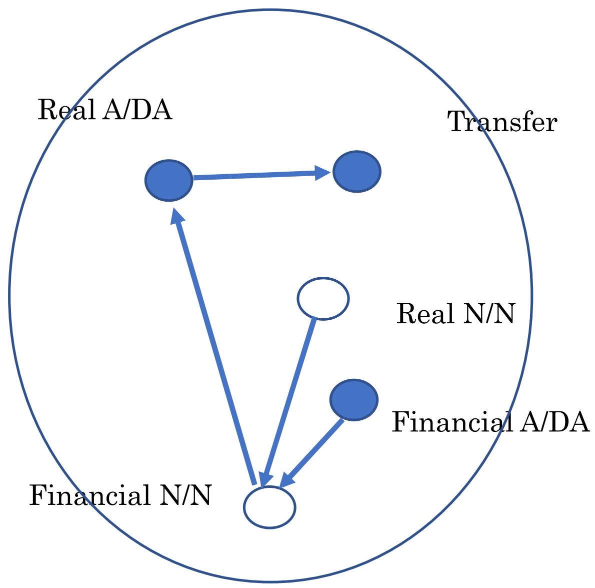

Figure 5: the sequence of accounts(vertices)

One additional point to be noted may be about the treatment of financial items. In the present approach, financial items appear when they are issued as a debt by some

[7] For each economic agent, there is only one transfer vertex and it is of A/DA type. It seems to be difficult to introduce the distinction between current transfers and capital transfers.

economic agent and disappear when an arc carrying the item reaches the original issuer's vertex. In other words, the number of the types of financial items are quite so many because they are classified by their issuers as well as their categorical properties. [8]

Here, the next aggregation step, the one over j rather than i can start. To do so, you need to define sectors, that is, groupings of economic agents. However, it may be worth noting that the aggregation over some subset of the index set for items (I) and the index set for agents (J). In fact, in the SNA1968, because of the so-called real-financial

dichotomy, the different sectoring principle can be applied to the real sphere of accounts and the financial sphere of accounts.

Finally in this section, some of the merits graph-theoretical approaches have may be mentioned. Firstly, educating students with no prior knowledge of business accounting or book keeping may be easy by using digraphs, remembering any digraph can be represented by a set of T forms. Secondly, as is well known, any digraph can be represented by an adjacency matrix and an incidence matrix so that it may be easy to store and operate national accounting data mathematically. Thirdly, graph theory has been making remarkable progress. So, national accountants can utilise the concepts and theorems developed in graph theory to make progress in their own field.

Finally, in contrast to the "cost-benefit" view of accounting (the vertical view of national accounting), the graph-theoretical approach naturally focuses on a "cooperation" view of the economy (the horizontal view of national accounting). In fact, this view reminds us of the substantive meaning of economics by Karl Polanyi.[9] He stated some 60 years ago in a well-known article that the substantive meaning of "economic" derives from man's dependence for his living upon nature and his fellows.

Concluding remarks

The Norwegian tradition of national accounting began with Ragnar Frisch's study of "eco-circ." His work was followed by Odd Aukrust's studies. He clarified set-theoretical relations between macro-accounting concepts by using an axiomatic approach.

Aukrust's contribution was followed by many attempts towards the axiomatisation of national accounting including Bénard [1972] and others. Furthermore, his work affected business accountants as well and the axiomatisation of business accounting by Richard

[8] Compare this treatment with those in Aukrust [1955/1998, 1960] and Deguchi [1988a,b 1989,1990, 2004].

[9] Polanyi [1957] p.243.

Mattessich [1956, 1964] was a typical example..

Hiroshi Deguchi also entered into this field of study and clarified an algebraic structure included in the bookkeeping system as an information base for an individual economic agent.

In line with Frisch's idea of ecocirc, Itsuo Sakuma axiomatised national accounting by using a graph-theoretical approach and concluded that national accounting is an Eulerisation of any digraph deemed to represent the economy.

What significance does the axiomatisation of national accounting have? To design national accounts system, you should recognise that there are fundamental requirements from the basic or axiomatic structure of national accounting. But as was suggested in the text for the graph-theoretical approach, there are some difficulties to define production axiomatically (only by using graph-theoretical terms). Note that once the concept of production is determined, the concept of capital as well as consumption can be determined. National accountants have their own analytical purposes and subject to the axiomatic requirements, they can define production and/or some other concepts in line with their own purposes.

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Appendix:

A formal presentation of the definitions and axioms: a graph-theoretical approach

Definition: The Economy

The Economy is a digraph G = (V, A, $\partial+$, $\partial-$), or G=(V,A) for short, where

V is a vertex set,

A is an arc set,

$\partial+$ is a map $A \to V$. For any arc $a \in A$, this map gives its initial vertex.

$\partial-$ is a map $A \to V$. For any arc $a \in A$, this map gives its terminal vertex.

Axiom 1: V is a (non-empty) finite set.
Axiom 2: A is a finite set.

Notations on the classification system of vertices:
Because the vertex set has various subsets, a triple subscript notation is used. So,

$$V_{ijk}$$

represents the element of the vertex set concerning the *i*-th item and the *j*-th economic agent ($i \in I, j \in J$) and the third subscript *k* denotes the type of the vertex, A/DA or N/N.

$$V = \bigcup_{\substack{i \in I \\ j \in J \\ k \in K}} V_{ijk}$$

The index set *J* may be subdivided into "sectors." Note that sectoring may be conducted for some subset of the index set *I* as well as for *I* as a whole.
The index set *I* may be subdivided into:
Goods (G)
Services (S)
Labour services (L)
Land (T) including forests
The above four subsets constitute Real (R)
Financial items (F)
Socially constructed rights (N) including copy rights, trademarks, etc.

The above categories may be further subdivided into various categories thereof.

V_{ijk} may not exist for some combinations of *i,j,k*. For example, services categories ($i \in S$) have no N/N type of vertices. However, even if they are, they are isolated by nature and will be neglected in what follows.

Axiom 3: There are "adjacency" rules for each vertex type as an initial vertex or as a terminal vertex. Admissible adjacency relations are as follows;

G(A/DA) → G(N/N)
G(N/N) → G(N/N), G(N/N) →G(A/DA)
S(A/DA) → S(A/DA)
L(A/DA) → L(A/DA)
T(N/N) →T(N/N)
F(A/AD) → F(N/N), F(N/N) → F(N/N)
$F(N/N)_j \rightarrow F(A/DA)_j$
A financial item issued by agent j as his/her debt disappears only at j' vertex(A/DA)
N(A/DA) g → N(N/N)
Nonfinancial intangible assets are socially constructed. The consolidated government (the general government plus central banks) sometimes issues special rights called copy rights, patent rights etc. to some economic agent, and there may be other intangible assets like trademarks registered or not, which are created by the society itself so to speak. The suffix *g* denotes the (consolidated) government (and the society, which is regarded as the issuing institution of these intangible assets including trademarks not registered.

Note: This rule is applicable only to the original digraph before aggregation.
Note: The property of arcs is determined by which vertex they depart at and which vertex they terminate into.

Note: It is already noticed in the text that separating production processes from the whole production/consumption process may be difficult. However, this separation is crucial to some economic analyses. The following postulates are some of the candidates that seem to be ways out.

Postulate 1: The set of the economic agents *J* should be able to be divided into that of producers and that of consumers. That is,

$$J = J_p \cup J_c, \quad J_p \cap J_c = \varnothing ,$$

where J_p is the subset of the index set *J* which is deemed to be producers, and J_c is the subset of the index set *J* which is deemed to be consumers. Furthermore, it is assumed that producers have the only production process (as understood to be a technical combination of inputs and outputs). Note that this process may be multi-output.

Postulate 2: The set of the index set of items G and S should be able to be divided into that of goods and services for production and that of goods and services for consumption, that is,

$$\begin{aligned} G = G_p \cup G_c, \quad G_p \cap G_c = \varnothing; \\ S = S_p \cup S_c, \quad S_p \cap S_c = \varnothing \end{aligned},$$

where the notations used are clear-cut. By aggregating vertices over G_c and S_c, you can get the consumption vertex. The remainder will be production vertices. As in the above, it is assumed that each economic agent has the only production process.

Under Postulate 1 or 2, for any j, the set of real A/DA vertices can be reorganised into the set of production vertices and the set of consumption vertices.

Note: These alternative postulates are necessary to define "capital." The reorganisation involved may be some aggregation procedure as seen above on some assumptions. However, it may not be the case.

Definition: National Accounts

When a digraph "the Economy" is Eulerised, it is called National Accounts.

An Example of Eulerisation

See text.