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Relative Bipolarization Orderings

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Abstract

Bipolarisation indices are helpful to quantify the relative presence or absence of a middle-class by measuring the degree to which distributions move away from equality toward a bimodality with clusters spreading further apart from each other. Relative bipolarisation indices have the advantage of being insensitive to changes in the unit of measurement and typically appear as explicit or implicit functions of income gaps from the median normalised by either the mean or the median itself. However the literature has not discussed the implications of alternative ways to construct indices based on these gaps. We introduce the properties of symmetry, bottom asymmetry and top asymmetry in order to elucidate the ethical implications of treating gaps below and above the median equally or differentially. Besides providing examples of symmetric and asymmetric indices in the literature, we identify a broad set of partial orderings pertaining to robust bipolarisation comparisons with broad subclasses of relative bipolarisation indices stemming from combinations of different (a)symmetry property fulfillments with alternative gap normalisation procedures. The ensuing stochastic dominance criteria are compared to existing proposals in the literature and their ordering power is compared studying household consumption in Georgia between 2009 and 2020.

Keywords: Bipolarization, middle class, stochastic dominance.

JEL Classification: D63, I3.

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Policy makers, social researchers, and citizens are largely concerned with the presumed decline of the middle class in high-income countries. Such a decline is usually regarded as a threat for social cohesion, democratic politics, welfare, and economic performance (e.g. Przeworski, 2004, Acemoglu and Robinson, 2006). The concern is significantly fueled by

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the observed increase in inequality throughout most developed countries since the 1980s (e.g. as documented inter alia by Piketty, 2017, Aktinson, 2018, Milanovic, 2018). However, inequality and bipolarization are different concepts (Wolfson, 1994, Chakravarty, 2009); the latter measuring the relative degree of cohesion around the middle vis-a-vis clustering around the distributional tails. In fact, the evidence of ongoing raising bipolarization in these countries is still limited.

Moreover, it is widely acknowledged that any social welfare index can be regarded as the subjective view of a social evaluator on some specific aspects of the observed distribution of well-being attributes. Hence, embedded in each index are value judgements that are likely to be debated. As a result, two different indices can yield opposite conclusions when applied to the same data, even if they comply with the same set of minimal desirable properties. To circumvent this issue whenever possible, the usual approach is deploying comparison criteria that identify the situations in which distribution-sensitive welfare indices based on the same axiomatic framework will produce the same ranking of observed distributions; that is, a robust ordering.

The bipolarisation literature features several proposals of criteria for the identification of robust comparisons in the form of stochastic dominance conditions. These usually relate to indices satisfying a minimum set of desirable properties, chiefly some versions of the two key transfer axioms: one stating that a bipolarisation index should increase whenever a regressive transfer occurs between two people on opposite sides of the median, and another demanding that the bipolarisation index increases whenever a progressive transfer takes place between two people on the same side of the median (either below or above) (Wang and Tsui, 2000, Duclos and Échevin, 2005, Bossert and Schworm, 2008, Chakravarty, 2009, Chakravarty and D'Ambrosio, 2010, Foster and Wolfson, 2010, Yalonetzky, 2014). Though definitely useful, we claim that most of these conditions are often too demanding and yield relatively highly incomplete (partial) orderings because they apply to broad classes of bipolarisation indices which, in turn, differ on relevant traits despite fulfilling the key axioms in some of their versions.[1]

Thus, we propose novel criteria for the identification of robust comparisons which incorporate additional noteworthy properties into the set of axioms that the related indices are expected to fulfill. To begin with, we focus on relative bipolarisation indices and discuss dominance criteria for indices normalised by the median and those normalised by the mean, separately, and highlighting that both classes relate to two different partial orderings. Additionally for each normalisation choice, we identify symmetric and asymmetric bipolarisation indices.[2] In the former, an income gap below the median and an income gap of the same magnitude above the median contribute to the index's value in the same

[1]We say "in some of their versions" because the two bipolarisation transfer axioms are not always written in the same manner. For instance, the spread axiom of Bossert and Schworm (2008) is different from Foster and Wolfson (2010)'s proposal. Likewise, the clustering axiom of Foster and Wolfson (2010) is different from Wang and Tsui (2000)'s.

[2]The term "symmetry" is generally used in a different way in the inequality-polarization-poverty-wellbeing literature. Indeed, it refers to the concept of horizontal equity that imposes the "equal treatment of the equals", that is the fact that two individuals with the same income should have the same contribution to the considered index, no matter their other (non-relevant) characteristics. Here, we prefer calling this property "anonymity" so as to avoid confusion with our own concept of symmetry.

measure. By contrast, in asymmetric bipolarisation indices the size of the contribution of income gaps depends on the income's position vis-a-vis the median. This distinction is relevant if we may want to exert differential judgement on clustering behaviour in the two halves of the distribution. For instance, we can use (what we call) bottom-asymmetric indices if we want our social evaluation to be relatively more sensitive to clustering among the poorest half, or top-asymmetric indices if, instead, we prefer highlighting clustering among the richest half.

In order to derive the dominance conditions and related partial orderings we start from two broad classes of rank-dependent and rank-independent relative bipolarisation indices. For the former, and considerig both mean and median normalisations, we identify four partial orderings: (1) a general one for the whole class; (2) one for a subclass of rank-dependent indices satisfying the symmetry property; (3) one for a subclass fulfilling bottom asymmetry; (4) and another one fulfilling top asymmetry. For rank-independent indices we find that it pays to derive dominance conditions for mean-normalised indices separately from median-normalised indices. For each subclass of rank-independent indices, again, we derive four sets of robustness conditions: (1) for each respective class as a whole; (2) for rank-independent indices satisfying symmetry; (3) for those fulfilling bottom asymmetry; and (4) those fulfilling top asymmetry. Notably, the partial orderings generated by the imposition of asymmetry properties can be tested with so-called sequential dominance conditions (Duclos and Makdissi, 2005, Bresson, Apablaza, and Yalonetzky, 2016).

We also identify the situations in which some of the new relative bipolarisation dominance criteria coincide with existing proposals in the literature (mainly those of Bossert and Schworm, 2008, Wang and Tsui, 2000, Foster and Wolfson, 2010, Yalonetzky, 2014). Then we test the empirical relevance of these conditions with a study of household consumption in Georgia between 2009 and 2020. Between the period's endpoints, the Caucasian country experienced a decline in the value of several inequality and bipolarisation indices. However, the year-to-year changes in indicators measuring both dispersion concepts have been anything but homogenous. Among other details, our robustness assessments show (as expected) that some consecutive-year pairwise comparisons of bipolarisation are only robust when are prepared to impose symmetry or asymmetry properties. Moroever, comparing the two endpoints we find that bipolarisation reduction in Georgia is robust with median-normalised indices or when mean-normalised indices are required to comply with top asymmetry, i.e. featuring higher bipolarisation when clustering develops among the top part of the distribution.

The rest of the paper proceeds as follows. Section 1 introduces the notation and the axiomatic framework underpinning relative bipolarization partial orderings, including the new axioms needed to formalise the distinctions between symmetric and asymmetric bipolarisation indices. Sections 2 and 3 derive the robustness criteria for subclasses of rank-dependent and rank-independent relative bipolarisation indices, respectively. Section 4 compares the partial orderings to those proposed in the literature. Section 5 provides the empirical illustration on Georgia. Finally the paper concludes in section 6 with some remarks.

1 Notation and axiomatic framework

Let $y \in \mathbb{R}_+$ be some well-being indicator such that well-being is a strictly increasing function of it. For the sake of simplicity, we will call that variable income, though it can be any non-negative continuous monetary or non-monetary well-being indicator.[3] We also assume the population is composed of n individuals, $n \in \mathbb{N}\backslash\{0\}$. Then the vector $\boldsymbol{y} := (y_1, \ldots, y_n)$ is the observed income distribution whose median and mean are $m_{\boldsymbol{y}}$ and $\mu_{\boldsymbol{y}}$, respectively. To ease the reading, the subscript on these two symbols will be dropped as long as this simplification does not cause any confusion. In all circumstances we rank incomes in ascending order such that y_i represents the ith lowest income. That way, we can conveniently partition $\boldsymbol{y}$ into two non-overlapping subsets $\boldsymbol{y_L}$ and $\boldsymbol{y_H}$, such that $\boldsymbol{y_L} := (y_1, y_2, \ldots, y_{\frac{n}{2}})$ and $\boldsymbol{y_H} := (y_{\frac{n}{2}+1}, y_{\frac{n}{2}+2}, \ldots, y_n)$.

A bipolarisation index Ψ is a mapping from $\mathbb{R}^n_+$ to $\mathbb{R}$ that is expected to comply with at least the following axioms:

Replication (POP): $\forall \boldsymbol{y} \in \mathbb{R}^n_+$, $\Psi(\boldsymbol{x}) = \Psi(\boldsymbol{y})$ if $\boldsymbol{x}$ is a k-fold replication of $\boldsymbol{y}$, and $k \in \mathbb{N}\backslash\{0\}$.

Anonymity (ANO): $\forall \boldsymbol{y} \in \mathbb{R}^n_+$, $\Psi(P\boldsymbol{y}) = \Psi(\boldsymbol{y})$ if P is an $n \times n$ permutation matrix.[4]

Spread-increasing transfer (SPR): $\forall \boldsymbol{y} \in \mathbb{R}^n_+$, $\Psi(\boldsymbol{x}) \geqslant \Psi(\boldsymbol{y})$ if $\boldsymbol{x}$ is obtained from $\boldsymbol{y}$ through a regressive transfer involving incomes i and j such that $y_i \in \boldsymbol{y_L}$ and $y_j \in \boldsymbol{y_H}$.

The first two axioms are widely used in distributional analysis. In the present context, they respectively make bipolarisation indices independent of the population size (implying that the index's value must be interpreted as a degree of bipolarisation, rather an amount) and non-relevant information regarding income earners.[5]

Increasing clustering transfer (ICT): $\forall \boldsymbol{y} \in \mathbb{R}^n_+$, $\Psi(\boldsymbol{x}) \geqslant \Psi(\boldsymbol{y})$ if $\boldsymbol{x}$ is a obtained from $\boldsymbol{y}$ through a progressive transfer of $\delta > 0$ involving incomes i and j such that either $x_i \leqslant x_j \leqslant m_{\boldsymbol{y}}$ or $m_{\boldsymbol{y}} \leqslant x_i \leqslant x_j$, and $m_{\boldsymbol{x}} = m_{\boldsymbol{y}}$.

The third axiom, ICT, is specific to bipolarization analysis and complements SPR in defining bipolarization as it is related to the impact of progressive transfers within the bottom or the top half of the income distribution. It essentially means that the convergence of incomes within either the top or the bottom half of the distribution towards their respective means strengthens bipolarity. ICT is the increased-bipolarity axiom suggested

[3]Assuming non-negative values for income is a rather conservative choice. Indeed, most absolute bipolarisation indices can be used with income taking values from the whole set of real numbers. In the context of relative bipolarisation, most indices only require the average income of the distribution's bottom half to be non-negative and the average income of the top half to be strictly positive.

[4]A square matrix is a permutation matrix if it is bistochastic with exactly one strictly positive value in each row and column.

[5]That is, equivalence scales are given the delicate role of integrating differences in needs in the analysis before aggregation is performed using the bipolarisation index.

by Wang and Tsui (2000) and Chakravarty and Majumder (2001). It is stronger than the version proposed in Foster and Wolfson (2010) and Bossert and Schworm (2008) as the latter does not impose the equality $m_{\boldsymbol{x}} = m_{\boldsymbol{y}}$. As noted by Yalonetzky (2017), satisfying the stronger version can be difficult with indices that use the median to normalise distances from the median as progressive transfers are likely to change the median's value.

Let $\mathbf{1}^n$ be a vector of ones of size n. For practical purposes, the following three axioms are also often imposed:

Normalisation (NOR): $\forall y \in \mathbb{R}_+$, $\Psi(y\mathbf{1}^n) = 0$.

Continuity (CON): Ψ is a continuous measure on $\mathbb{R}^n_+$.

Scale invariance (SCI): $\Psi(\lambda \boldsymbol{y}) = \Psi(\boldsymbol{y})\ \forall \boldsymbol{y} \in \mathbb{R}^n_+$, $\lambda \in \mathbb{R}_{++}$.

NOR simply claims that the bipolarisation index should return a 0 value when the distribution is perfectly egalitarian; that is, in the complete absence of bipolarisation. CON is generally imposed in order to avoid marginal errors in the measurement of individual income having non-marginal effects on the estimated level of bipolarization. This justification, however, conceals the normative implications of the axiom. Indeed, if we regard bipolarization indices as the expression of possibly implicit Bergson-Samuelson social welfare functions, CON means that such functions must be continuous with respect to individual incomes, hence precluding the possibility of thresholds effects both from the social and the individual point of view.

Scale invariance is usually supported whenever indices are demanded to be independent of the monetary unit chosen for measuring incomes. However, it is well-known (Kolm, 1976) that scale invariance is not as neutral as it seems at first sight since it may be imposed for the two following reasons: *(i)* making sure that orderings are preserved after changes in the income measurement unit and *(ii)* determining how an additional income should be distributed among individuals in order to preserve the initial level of bipolarity. If one is only concerned with the first issue, then a unit consistency axiom, first introduced in inequality measurement by Zheng (2007), shall be preferred. In the context of bipolarization measurement, this axiom becomes:

Unit consistency (UNC): $\forall \{\boldsymbol{x}, \boldsymbol{y}\} \subset \Re^n_+$, $\lambda \in \mathbb{R}_{++}$, $\Psi(\lambda \boldsymbol{x}) > \Psi(\lambda \boldsymbol{y})$, if and only if $\Psi(\boldsymbol{x}) > \Psi(\boldsymbol{y})$.

Once this axiom is imposed, scale invariance only adds that equally proportional increases in all incomes preserve the level of bipolarity. It is worth noting that the use of scale-invariant indices generally means that these indices are relative indices. Distances with respect to the median are then expressed as ratios with respect to some reference income level for the computation of the index. Most relative bipolarization indices used the median to normalize distances, like the Foster and Wolfson's (2010) index:

$$\Psi^{FW}(\boldsymbol{y}) = \frac{\big(1 - G(\boldsymbol{y}^H)\big)\mu_{\boldsymbol{y}^H} - \big(1 + G(\boldsymbol{y}^L)\big)\mu_{\boldsymbol{y}^L}}{2m_{\boldsymbol{y}}}, \tag{1}$$

where G is the Gini coefficient. However this specific choice of normalization may ultimately be debatable since, to the best of our knowledge, no justification has ever been provided for it. One can easily prove that normalizing distances by any other quantile, or by the mean, also enable proposing relative bipolarization indices that comply with the weakest versions of the aforementioned axioms. For instance, Rodríguez and Salas (2003) propose a mean-normalized version of Foster and Wolfson's (2010) index that also satisfies POP, ANO, SPR, ICT, NOR, and SCI:

$$\Psi^{RS}(\boldsymbol{y}) = \frac{\big(1 - G(\boldsymbol{y}^H)\big)\mu_{\boldsymbol{y}^H} - \big(1 + G(\boldsymbol{y}^L)\big)\mu_{\boldsymbol{y}^L}}{2\mu_{\boldsymbol{y}}}. \tag{2}$$

The choice of the normalization factor can often be justified by the comparison with reference situations that are associated either with a minimum or a maximum value for the relative distributional index. For instance, in the case of poverty measurement, dividing income gaps by the value of the poverty line makes sense because it is generally assumed that the level of poverty would be at its minimum in a situation where everyone in the population had an income at least equal to this threshold. Using ratios with respect to the poverty line then makes it possible to assess how far the economy is from this zero-poverty target. Considering relative bipolarization, NOR claims that the minimum level is reached when everyone has the same income level. But this benchmark situation is not helpful for the choice of normalisation factor since all quantiles and mean income are identical in a situation of perfect equality.[6]

The opposite extreme situation, namely maximum bipolarity, is observed when those in the bottom 50% have zero income while everyone in the top half enjoys the same strictly positive income level. In this case, the choice of a specific quantile may be decisive, even though several quantiles are identical and, chiefly, the mean and the median are equal.

Sometimes, the very definition of the distributional concept immediately settles any discussions regarding the choice of normalization factor. This is, for instance, the case of relative inequalities defined as the dispersion of income shares in total income, so that normalization with respect to mean income is straightforward. In the case of relative bipolarization, existing definitions provide generally little guidance concerning this specific point. An exception is Rodríguez and Salas (2003) who define their relative bipolarization measures relying on the (relative) Lorenz curve, hence implying mean-normalization. Foster and Wolfson (2010) also connect relative bipolarization to characteristics of the Lorenz curve for decomposition purposes, but propose a median-normalized index.

For the rest of the paper, we opt for considering only median- and mean-normalization, keeping in mind that other reference distributional standards (e.g. other quantiles) could be consistent with relative bipolarization measurement. However, we acknowledge that no decisive argument can be given at that point regarding the most appropriate way to normalize distances from the median for relative bipolarization evaluation. Sections 2

[6]Note that this is also the case for relative inequality measures. However, as stressed in the next paragraph, mean-normalization is a straightforward choice once relative inequality is defined as the dispersion of income shares.

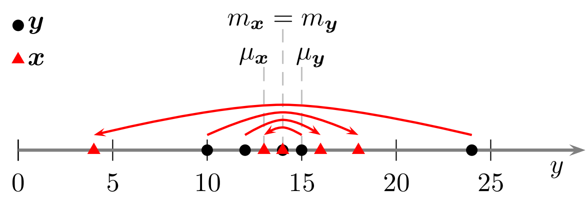


Figure 1: m-type symmetry.

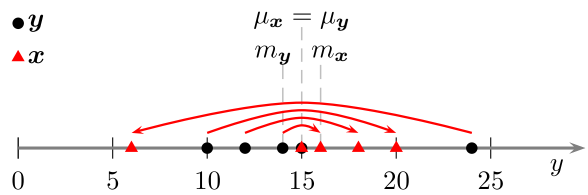


Figure 2: μ-type symmetry.

and 3 as well as the empirical illustration in section 5 will show the importance of this choice.

Now we introduce a novel set of axioms in order to study the influence of the incomes' observed distances from their median on the level of bipolarity. More precisely for instance, one could demand from relative bipolarization indices Ψ the same sensitivity to income changes in the bottom half as to income changes in the top half of the distribution. This property shall be observed for instance if, starting from a symmetric income distribution, a progressive transfer $\delta > 0$ from $x_i = m - a$ to $x_j = m - b$ (with $b > a > 0$) induces the same change in Ψ as if the same amount had been transferred from $x_{j'} = m + b$ to $x_{i'} = m + a$. This behaviour is notably apparent in the class of relative bipolarization indices based on the distribution of absolute deviations from the median (Duclos and Échevin, 2005). More generally for any possible distribution, this indifference to the location of distances with respect to the median, or the mean, is captured in its weakest forms by the following two axioms, respectively:

m-type symmetry (m-SYM): $\forall \{\boldsymbol{x}, \boldsymbol{y}\} \subset \mathbb{R}^n_+$, $\Psi(\boldsymbol{x}) = \Psi(\boldsymbol{y})$ where $\boldsymbol{x} = \boldsymbol{y} + 2(m_{\boldsymbol{y}} - \boldsymbol{y})$.[7]

μ-type symmetry (μ-SYM): $\forall \{\boldsymbol{x}, \boldsymbol{y}\} \subset \mathbb{R}^n_+$, $\Psi(\boldsymbol{x}) = \Psi(\boldsymbol{y})$ where $\boldsymbol{x} = \boldsymbol{y} + 2(\mu_{\boldsymbol{y}} - \boldsymbol{y})$.[8]

The transformations involved in each of the two axioms are illustrated on figures 1 and 2 with $\boldsymbol{x} = (10, 12, 14, 15, 24)$. In both cases, the axioms state that flipping income values around a certain measure of central tendency (the median or the mean) while preserving absolute distances from the same distributional standard, also preserves relative bipolarization. For m-SYM, the reference distributional standard is the median, while the mean is used for μ-SYM. In the general case where $m_{\boldsymbol{y}} \neq \mu_{\boldsymbol{y}}$, the transformations either imply a change in the mean (m-SYM) or in the median (μ-SYM). Indeed with m-SYM, the equally-bipolarized distribution is $(4, 13, 14, 16, 18)$), while for μ-SYM the equally-bipolarized distribution is $(6, 15, 16, 18, 20)$. It can easily be seen that there is no simple set

[7]Clearly, $\boldsymbol{x} = \boldsymbol{y} + 2(m_{\boldsymbol{y}} - \boldsymbol{y})$ stems from $\boldsymbol{x} - m_{\boldsymbol{y}} = m_{\boldsymbol{y}} - \boldsymbol{y}$.
[8]Clearly, $\boldsymbol{x} = \boldsymbol{y} + 2(\mu_{\boldsymbol{y}} - \boldsymbol{y})$ stems from $\boldsymbol{x} - \mu_{\boldsymbol{y}} = \mu_{\boldsymbol{y}} - \boldsymbol{y}$.

of operations (e.g. multiplication by a scalar) to obtain the former distribution from the latter, which underlines the normative difference between m-SYM and μ-SYM.

Examples of indices satisfying the symmetry axioms include Ψ^{FW} (respecting m-SYM) and Ψ^{RS} (respecting μ-SYM). This is also the case of the class of relative indices studies in Duclos and Échevin (2005) as well as of the fourth class of indices considered by Wang and Tsui (2000).

Meanwhile, several examples of relative bipolarization indices not fulfilling this symmetry property appear in the second class of bipolarization indices proposed by Wang and Tsui (2000):

$$\Psi^{WT}_{\boldsymbol{\alpha}}(\boldsymbol{y}) = \frac{1}{n}\sum_{i=1}^{n}\alpha_i\left|\frac{m_{\boldsymbol{y}} - y_i}{m_{\boldsymbol{y}}}\right|, \tag{3}$$

with $\alpha_i > \alpha_{i-1} > 0\ \forall i$ such that $y_i < m_{\boldsymbol{y}}$, and $\alpha_i > \alpha_{i+1} > 0\ \forall i$ such that $y_i > m_{\boldsymbol{y}}$. $\Psi^{WT}_{\boldsymbol{\alpha}}$ complies with m-SYM only when $\alpha_i = \alpha_{n-i+1}\ \forall i \in \{1, \dots n\}$. This last example illustrates how the set of relative bipolarization indices shrinks once one of these symmetry properties is imposed. In addition to the distinction between symmetric and asymmetric bipolarisation measures, we can also delve deeper into different forms of asymmetry by considering intermediate classes of indices whose sensitivity to increasing clustering transfers is systematically more pronounced in a particular part of the distribution. Thus, depending on the normalization procedure used in order to satisfy scale invariance, we can propose the two following sets of asymmetry axioms:

m-type bottom (resp. top) asymmetry (m-B-ASYM [resp. m-T-ASYM):] $\forall\{\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'\} \subset \mathbb{R}^n_+$ such that $\boldsymbol{x} = \boldsymbol{y} + 2(m_{\boldsymbol{y}} - \boldsymbol{y})$, $\boldsymbol{x}' = \boldsymbol{y}' + 2(m_{\boldsymbol{y}'} - \boldsymbol{y}')$, if $\boldsymbol{x}'$ is obtained from $\boldsymbol{x}$ through a progressive transfer within $\boldsymbol{x}_L$ (resp. $\boldsymbol{x}_H$) and $m_{\boldsymbol{x}'} = m_{\boldsymbol{x}}$, then $\Psi(\boldsymbol{x}') - \Psi(\boldsymbol{x}) \geqslant \Psi(\boldsymbol{y}') - \Psi(\boldsymbol{y})$.

μ-type bottom (resp. top) asymmetry (μ-B-ASYM [resp. μ-T-ASYM):] $\forall\{\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'\} \subset \mathbb{R}^n_+$ such that $\boldsymbol{x} = \boldsymbol{y} + 2(\mu_{\boldsymbol{y}} - \boldsymbol{y})$, $\boldsymbol{x}' = \boldsymbol{y}' + 2(\mu_{\boldsymbol{y}'} - \boldsymbol{y}')$, if $\boldsymbol{x}'$ is obtained from $\boldsymbol{x}$ through a progressive transfer within $\boldsymbol{x}_L$ (resp. $\boldsymbol{x}_H$) and $m_{\boldsymbol{x}'} = m_{\boldsymbol{x}}$, then $\Psi(\boldsymbol{x}') - \Psi(\boldsymbol{x}) \geqslant \Psi(\boldsymbol{y}') - \Psi(\boldsymbol{y})$.

Starting from a non-degenerate symmetric income distribution, both bottom- (alternatively top-) asymmetry axioms mean that a median-preserving progressive transfer within the poorest (alternatively richest) half of the population will raise bipolarity more than a symmetric change among the richest (alternatively poorest) half. Applying this reasonning recursively, one can easily show that these axioms convey a preference for negatively skewed distributions when compared with the distribution obtained from a symmetric transpose around the median (for m-B-ASYM) or the mean (for μ-B-ASYM). Preference for positively skewed distribution is obtained with m-T-ASYM or μ-T-ASYM. For instance, with indices $\Psi^{WT}_{\boldsymbol{\alpha}}$, axiom m-B-ASYM is fulfilled if and only if $\alpha_i - \alpha_{i-1} \geqslant \alpha_{n-i} - \alpha_{n-i+1}\ \forall i$ such that $y_i \leqslant m_{\boldsymbol{y}}$.

Another example is the following generalization of a class of indices proposed by Chakravarty

(2009, chapter 4):

$$\Psi^C_{\varepsilon,\alpha}(\boldsymbol{y}) = \frac{1}{m_{\boldsymbol{y}}}\left(\frac{1}{n}\left((1+\alpha)\sum_{y_i\in\boldsymbol{y}_L}|m_{\boldsymbol{y}}-y_i|^{\varepsilon} + (1-\alpha)\sum_{y_i\in\boldsymbol{y}_H}|m_{\boldsymbol{y}}-y_i|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}}, \qquad (4)$$

with $\varepsilon \in]0,1[$ and $\alpha \in [0,1[$ to comply with m-B-ASYM, or $\alpha \in]-1,0]$ to satisfy m-T-ASYM instead.

2 Dominance criteria: rank-dependent indices

Classifying relative bipolarisation indices according to their dependence (or lack thereof) on income ranks is useful for the derivation of robust ethical criteria for the bipolarisation comparisons. Moreover, we can always combine the two sets of criteria to deduce new criteria.

For any distribution $\boldsymbol{y}$, let $y(p) : [0,1] \to \mathbb{R}_+$ be the quantile function. We first consider a general class of rank-dependant bipolarization indices Ψ^r of the form:

$$\Psi^r(\boldsymbol{y}) := \int_0^{0.5} w_L(p)g_L(p)\,dp + \int_{0.5}^{1} w_H(p)g_H(p)\,dp \qquad (5)$$

where $g_L(p) := \frac{m_{\boldsymbol{y}}-y(p)}{m_{\boldsymbol{y}}}$ or $\frac{m_{\boldsymbol{y}}-y(p)}{\mu_{\boldsymbol{y}}}$ and $g_H(p) := \frac{y(p)-m_{\boldsymbol{y}}}{m_{\boldsymbol{y}}}$ or $\frac{y(p)-m_{\boldsymbol{y}}}{\mu_{\boldsymbol{y}}}$ depending on the choice of normalization variable. The indices in (5) were first introduced by Wang and Tsui (2000, Proposition 4) (considering median normalization) and comply with axiom SPR when $w_L(p) \geqslant 0$ and $w_H(p) \geqslant 0$.[9] Moreover, they satisfy axiom ICT when $w_L(p) - w_L(q) \leqslant 0$ for any $p < q < 0.5$ and $w_H(p) - w_H(q) \geqslant 0$ for any $0.5 < p < q$, i.e. if gaps g_L and g_H are given more weights the closer incomes lie to the median. Let $\mathcal{B}_r$ be the class of indices satisfying all these requirements.

Let $G_L(p) := \int_p^{0.5} g_L(t)\,dt$ and $G_H(p) := \int_{0.5}^{p} g_H(t)\,dt$. Likewise, define $\Delta\Psi^r(\boldsymbol{y},\boldsymbol{x}) \equiv \Psi^r(\boldsymbol{y}) - \Psi^r(\boldsymbol{x})$ (and use the same Δ notation for other statistics). Then theorem 1 provides the dominance condition ensuring robustness of bipolarisation comparisons to any alternative choice of rank-dependent bipolarisation index in class $\mathcal{B}_r$:

Theorem 1. $\Delta\Psi^r(\boldsymbol{y},\boldsymbol{x}) \geqslant 0\ \forall\Psi^r \in \mathcal{B}_r$ *if and only if* $\Delta G_L(p) \geqslant 0\ \forall p \in [0, 0.5]$ *and* $\Delta G_H(p) \geqslant 0$ $\forall p \in [0.5, 1]$.

Proof. See appendix A. ■

Essentially, theorem 1 states that $\boldsymbol{y}$ is robustly more bipolarised than $\boldsymbol{x}$, according to any index in $\mathcal{B}_r$, if and only if the normalised income gaps of $\boldsymbol{y}$ general-Lorenz dominate those of $\boldsymbol{x}$ *above the median* and the normalised income gaps of $\boldsymbol{y}$ general-Lorenz dominate those of $\boldsymbol{x}$ *below the median*. Additionally, note that, unless both distributions feature

[9]See also Makdissi and Mussard (2011) for a study of indices Ψ^r.

means identical to the median, the robust ranking provided by theorem 1 will depend on the gap normalisation choice.

Furthermore, whenever $m_{\boldsymbol{y}} = m_{\boldsymbol{x}} = m$, then $m\Delta G_L(p) = \int_p^{0.5}[x(t) - y(t)]\,dt$ and $m\Delta G_H(p) = \int_{0.5}^p[y(t) - x(t)]\,dt$ for the case of median-normalised gaps. That is, the dominance conditions boil down to generalised-Lorenz domination of $\boldsymbol{y}$ over $\boldsymbol{x}$ *above the median* and to inverse-generalized-Lorenz dominance *below the median*.[10] A similar result ensues for gaps normalised by the mean when both $m_{\boldsymbol{y}} = m_{\boldsymbol{x}}$ and $\mu_{\boldsymbol{y}} = \mu_{\boldsymbol{x}}$.

2.1 Asymmetric indices

We now consider the subclass of rank-dependent bottom-asymmetric indices $\mathcal{B}_{rL} \subset \mathcal{B}_r$ such that $w_L(p) \geqslant w_H(1-p)$ and $w_L(p) - w_L(q) \geqslant w_H(1-p) - w_H(1-q)$ $\forall 0 \leqslant p < q \leqslant 0.5$. These two conditions ensures rank-dependent indices Ψ^r satisfy either m-B-ASYM or μ-B-ASYM. Theorem 2 provides the dominance condition ensuring robustness of bipolarisation comparisons to any alternative choice of rank-dependent bipolarisation index in class $\mathcal{B}_{rL}$:

Theorem 2. *$\Delta\Psi^r(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0$ $\forall \Psi^r \in \mathcal{B}_{rL}$ if and only if $\Delta G_L(p) \leqslant 0$ $\forall p \in [0, 0.5]$ and $\Delta\big(G_L(1-p) + G_H(p)\big) \geqslant 0$ $\forall p \in [0.5, 1]$.*

Proof. See appendix A. ■

Note that $G_L(1-p) + G_H(p) = \int_{0.5}^p \frac{y(t)-y(1-t)}{m_{\boldsymbol{y}}}dt$ or $G_L(1-p) + G_H(p) = \int_{0.5}^p \frac{y(t)-y(1-t)}{\mu_{\boldsymbol{y}}}dt$ (depending on the normalisation choice), where $y(t) - y(1-t)$ is the income gap between one income above the median and one below such that both incomes are rank-equidistant from the median (thus, e.g. for $t = 0.5$ we get $y(0.5) - y(0.5) = 0$ and for $t = 1$ we get the maximum income gap from any pair in the population, namely $y(1) - y(0)$). Then theorem 2 states that $\boldsymbol{y}$ is robustly more bipolarised than $\boldsymbol{x}$, according to any bottom-asymmetric rank-dependent index in $\mathcal{B}_{rL}$, if and only if the normalised income gaps of $\boldsymbol{y}$ general-Lorenz dominate those of $\boldsymbol{x}$ *below the median* and the normalised difference of rank-equidistant income pairs in $\boldsymbol{y}$ general-Lorenz dominate those of $\boldsymbol{x}$.

As is to be expected due to the imposition of additional axioms, the dominance condition put forward by theorem 2 is less stringent than theorem 1's and relates to a more complete partial ordering. Meanwhile a similar result also exists for top-asymmetry. In this case we can consider a subclass of rank-dependent indices $\mathcal{B}_{rH} \subset \mathcal{B}_r$ characterised by $w_L(p) \leqslant w_H(1-p)$ and $w_L(p) - w_L(q) \leqslant w_H(1-p) - w_H(1-q)$ $\forall 0 \leqslant p < q \leqslant 0.5$ in order to fulfill either m-T-ASYM or μ-T-ASYM. Then the analogue of theorem 3 is:

Theorem 3. *$\Delta\Psi^r(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0$ $\forall \Psi^r \in \mathcal{B}_{rH}$ if and only if $\Delta G_H(p) \geqslant 0$ $\forall p \in [0.5, 1]$ and $\Delta\big(G_L(1-p) + G_H(p)\big) \geqslant 0$ $\forall p \in [0.5, 1]$.*

[10] Let $\mathcal{L}_y(p) = \int_0^p y(t)dt$ for any $p \in [0, 1]$ be the generalised Lorenz curve, then the inverse generalised Lorenz curve is defined by $\mathcal{IL}_y(p) = \int_0^p y(1-t)dt$ for any $p \in [0, 1]$. That is, the construction of the inverse-generalised Lorenz curve requires sorting incomes in descending order and adding them up accordingly, starting with the highest income. Similar definitions apply to Lorenz and inverse Lorenz curves.

Proof. Similar to proof of theorem 2 in appendix A. Left to the readers or available upon request. ■

Note, interestingly, that the dominance condition associated with top-asymmetric rank-dependent indices also features the general-Lorenz comparison of normalised gaps of rank-equidistant income pairs, but now a general-Lorenz comparison of normalised gaps *above the median* replaces the previous general-Lorenz comparison *below the median* in theorem 2.

2.2 Symmetric indices

Finally, rank-dependent indices complying either with m-SYM or μ-SYM must feature $w_L(p) = w_H(1-p)\ \forall p \in [0, 0.5]$. Let $\mathcal{B}_{rS} \subset \mathcal{B}_r$ be the corresponding subclass of symmetric rank-dependent bipolarization indices. Theorem 4 provides the corresponding dominance condition:

Theorem 4. $\Delta\Psi^r(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0\ \forall \Psi^r \in \mathcal{B}_{rS}$ *if and only if* $\Delta\big(G_L(1-p) + G_H(p)\big) \geqslant 0\ \forall p \in [0.5, 1]$.

Proof. See appendix A. ■

Again, the dominance condition of theorem 4 less demanding than theorem 1's and corresponds to a more complete partial ordering representing the agreement of all rank-dependent symmetric bipolarisation indices.

3 Dominance criteria: rank-independent indices

Following the aforementioned classification, we now consider robustness criteria for rank-independent indices. However, unlike their rank-dependent counterparts, we need to address median-normalized and mean-normalized indices separately, as the corresponding classes are slightly different.

3.1 Median-normalized indices

We consider bipolarisation indices that can be expressed as monotonic transformations of additively decomposable indices that only take into account the relative position of each income with respect to the median. Therefore, bipolarisation comparisons can be performed directly with the additively decomposable indices (that is, we can take the inverse of the monotonic transformations and end up with simple additively decomposable indices satisfying anonymity and population principle).

Let $\boldsymbol{y}_L^* := \big(\min\{y_1, m_{\boldsymbol{y}}\}, \dots \min\{y_n, m_{\boldsymbol{y}}\}\big)$ and $\boldsymbol{y}_H^* := \big(\max\{y_1, m_{\boldsymbol{y}}\}, \dots \max\{y_n, m_{\boldsymbol{y}}\}\big)$ The distribution $\boldsymbol{y}$ can be split into two parts, namely $\boldsymbol{y}_L$ and $\boldsymbol{y}_H$ that can respectively be described by the following cumulative distribution functions $F(y;\boldsymbol{y}_L^*)$ and $F(y;\boldsymbol{y}_H^*)$. Then we can consider bipolarisation indices $\Psi(\boldsymbol{y})$ of the form:

$$\Psi_m(\boldsymbol{y}) = \int_0^{\omega^+} \psi_L(y, m_{\boldsymbol{y}})\, dF(y;\boldsymbol{y}_L^*) + \int_0^{\omega^+} \psi_H(y, m_{\boldsymbol{y}})\, dF(y;\boldsymbol{y}_H^*), \tag{6}$$

The functional form in (6) ensures that Ψ_m satisfies both the population principle and the anonymity axiom. Normalization is met whenever $\psi_L(m_{\boldsymbol{y}}, m_{\boldsymbol{y}}) = \psi_H(m_{\boldsymbol{y}}, m_{\boldsymbol{y}}) = 0$. Let ψ_k' and ψ_k'', $k \in \{L, H\}$, respectively be the first-order and second-order derivatives of ψ_k with respect to y. Satisfaction of the SPR axiom requires $\psi_L'(y, m_{\boldsymbol{y}}) \leqslant 0$ and $\psi_H'(y, m_{\boldsymbol{y}}) \geqslant 0$ while $\psi_L''(y, m_{\boldsymbol{y}}) \leqslant 0$ and $\psi_H''(y, m_{\boldsymbol{y}}) \leqslant 0$ are needed for Ψ to fulfill the ICT axiom. Consistency with the concept of polarization also requires ψ_L (resp. ψ_H) to be a non-decreasing (resp. non-increasing) function of $m_{\boldsymbol{y}}$. Finally, we also impose functions ψ_L and ψ_H to be homogeneous of degree 0 with respect to y and $m_{\boldsymbol{y}}$. This condition is consistent with the use of scale invariant relative indices in which normalization is performed using the median. The class of indices of the form (6) satisfying all these properties is $\mathcal{B}_m$.

Assuming $\mu_{\boldsymbol{y}} > 0$, the homogeneity condition means that $\psi_k(y, m_{\boldsymbol{y}}) = \psi_k\left(\frac{y}{m_{\boldsymbol{y}}}\right)$, $k \in \{L, H\}$, so that (6) can indifferently be estimated using the distribution of $z = \frac{y}{m_{\boldsymbol{y}}}$.[11] Let $[0, z^+]$ be the domain of definition for z, and $\boldsymbol{z}_L^*$ and $\boldsymbol{z}_H^*$ be the respective censored versions of $\frac{\boldsymbol{y}}{m_{\boldsymbol{y}}}$, that is $\boldsymbol{z}_L^* := \frac{\boldsymbol{y}_L^*}{m_{\boldsymbol{y}}}$ and $\boldsymbol{z}_H^* := \frac{\boldsymbol{y}_H^*}{m_{\boldsymbol{y}}}$. Then (6) can be equivalently expressed as:

$$\Psi_m(\boldsymbol{y}) = \int_0^{z^+} \psi_L(z)\, dF(z;\boldsymbol{z}_L^*) + \int_0^{z^+} \psi_H(z)\, dF(z;\boldsymbol{z}_H^*), \tag{7}$$

To introduce our next result, it is necessary first to define $F^{(2)}(z;\boldsymbol{z}) := \int_0^z F(t;\boldsymbol{z})\, dt$ and $\bar{F}^{(2)}(z;\boldsymbol{z}) := \int_z^{z^+} \bar{F}(t;\boldsymbol{z})\, dt$ where $\bar{F}$ is the survival function. We then obtain theorem 5: :

Theorem 5. $\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0\ \forall \Psi_m \in \mathcal{B}_m$ *if and only if* $\Delta\bar{F}^{(2)}(z;\boldsymbol{z}_L^*) \leqslant 0\ \forall z \in [0, 1]$ *and* $\Delta F^{(2)}(z;\boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, z^+]$.

Proof. See appendix A. ∎

Theorem 5 states that $\boldsymbol{y}$ is robustly more bipolarised than $\boldsymbol{x}$, according to any index in $\mathcal{B}_m$, if and only if the median-normalised distribution of $\boldsymbol{y}$ general-Lorenz dominates $\boldsymbol{x}$'s *above the median* and the median-normalised distribution of $\boldsymbol{x}$ general-Lorenz dominates $\boldsymbol{y}$'s *below the median*. Furthermore, when $m_{\boldsymbol{y}} = m_{\boldsymbol{x}}$ the general-Lorenz conditions apply directly to the raw income distributions.

3.1.1 Asymmetric indices

Imposing the m-B-ASYM axiom means $|\psi_L'(y, m_{\boldsymbol{y}})| \geqslant \psi_H'(2m_{\boldsymbol{y}} - y, m_{\boldsymbol{y}})$ and $\psi_L''(y, m_{\boldsymbol{y}}) \leqslant \psi_H''(2m_{\boldsymbol{y}} - y, m_{\boldsymbol{y}})$, with both inequalities reversed in the case of m-T-ASYM. Let $\mathcal{B}_{mL} \subset \mathcal{B}_m$ be the corresponding subset of bipolarization indices satisfying m-B-ASYM. Theorem 6

[11] For the homogeneity result, see for instance Aczél (1966, section 5.2).

provides the robustness condition for rank-independent bottom-asymmetric bipolarisation indices:

Theorem 6. $\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0\ \forall \Psi_m \in \mathcal{B}_{mL}$ *if and only if* $\Delta\bar{F}^{(2)}(z; \boldsymbol{z}_L^*) \leqslant 0\ \forall z \in [0, 1]$, $\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}_L^*) + \Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, 2]$, *and* $\Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [2, z^+]$.

Proof. See appendix A. ■

Though less intuitive, the dominance conditions of theorem 6 are clearly less demanding than those of theorem 5 (as expected). Note that the general-Lorenz comparison of median-normalised distributions above the median now only needs to be conducted above twice the median. Instead, $\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}_L^*) + \Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0$ needs to be checked for incomes between the median and twice its value. Moreover, this condition is easier to meet than $\Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, 2]$ (involved in theorem 5) if the first condition, namely $\Delta\bar{F}^{(2)}(z; \boldsymbol{z}_L^*) \leqslant 0\ \forall z \in [0, 1]$, is already satisfied. This is a typical situation of so-called sequential dominance procedures. Accordingly, the partial ordering induced by theorem 6 is less incomplete.

In the case of m-T-ASYM we obtain the following theorem:

Theorem 7. $\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0\ \forall \Psi_m \in \mathcal{B}_{mL}$ *if and only if* $\Delta\bar{F}^{(2)}(0; \boldsymbol{z}_L^*) \leqslant 0$, $\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}_L^*) + \Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, 2]$, *and* $\Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, z^+]$.

Proof. Similar to proof of theorem 6 in appendix A. Left to the readers or available upon request. ■

The conditions of theorem 7 are quite similar to those of 6, with the main differences being that the general-Lorenz comparison of median-normalised distributions above the median is restored, but now the general-Lorenz comparison below the median only needs to be performed at $z = 0$.

3.1.2 Symmetric indices

The class of rank-independent indices in (6) satisfies m-SYM whenever $\psi_L(y, m_{\boldsymbol{y}}) = \psi_H(2m_{\boldsymbol{y}} - y, m_{\boldsymbol{y}})$. The resulting subclass is $\mathcal{B}_{mS} \subset \mathcal{B}_m$ and corresponds to the fourth class of bipolarization indices studies in Wang and Tsui (2000). The following dominance conditions can then be used to assess the robustness of bipolarization orderings for members of this subclass:

Theorem 8. $\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0\ \forall \Psi_m \in \mathcal{B}_{mS}$ *if and only if* $\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}_L^*) + \Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, 2]$, *and* $\Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [2, z^+]$.

Theorem 8 features two dominance conditions that need to be fulfilled in order to secure agreement among all rank-independent symmetric bipolarisation indices in the subclass $\mathcal{B}_{mS}$. Noticeably, when the two compared distributions are themselves symmetric only one condition becomes relevant, i.e. $\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}_L^*) + \Delta F^{(2)}(z; \boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1, 2]$.

3.2 Mean-normalized indices

We consider a class of mean-normalized and rank-independent relative bipolarization indices based on absolute income gaps from the median. To distinguish gaps related to relatively low income from those corresponding to relatively high incomes, let $\boldsymbol{g}_L^* = \left(\max\left\{\frac{m_{\boldsymbol{y}}-y_1}{\mu_{\boldsymbol{y}}},0\right\},\ldots\max\left\{\frac{m_{\boldsymbol{y}}-y_n}{\mu_{\boldsymbol{y}}},0\right\}\right)$ and $\boldsymbol{g}_H^* = \left(\max\left\{\frac{y_1-m_{\boldsymbol{y}}}{\mu_{\boldsymbol{y}}},0\right\},\ldots\max\left\{\frac{y_n-m_{\boldsymbol{y}}}{\mu_{\boldsymbol{y}}},0\right\}\right)$. Assuming relative income gaps are defined over the interval $[0,g^+]$, the considered indices are then of the form:

$$\Psi_\mu(\boldsymbol{y}) = \int_0^{g^+} \theta_L(g)\, dF(g;\boldsymbol{g}_L^*) + \int_0^{g^+} \theta_H(g)\, dF(g;\boldsymbol{g}_H^*) \tag{8}$$

with $\theta_k : [0,g^+] \to \mathbb{R}\ \forall k \in \{L,H\}$. It can easily be checked that the population principle, anonymity and scale invariance axioms are necessarily fulfilled. Normalization requires $\theta_L(0) = \theta_H(0) = 0$. The axiom SPR is satisfied whenever $\theta'_L(g) \geqslant 0$ and $\theta'_H(g) \geqslant 0\ \forall g[0,g^+]$. The fulfilment of axiom ICT requires $\theta''_L(g) \leqslant 0$ and $\theta''_H(g) \leqslant 0\ \forall g[0,g^+]$. Let $\mathcal{B}_\mu$ be the class of indices (8) satisfying all these requirements. We then have:

Theorem 9. $\Delta\Psi_\mu(\boldsymbol{y},\boldsymbol{x}) \geqslant 0\ \forall \Psi_\mu \in \mathcal{B}'_\mu$ *if and only if* $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) \leqslant 0$ *and* $\Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0$ $\forall g \in [0,g^+]$.

Proof. See appendix A. ■

Theorem 9 states that $\boldsymbol{y}$ is robustly more bipolarised than $\boldsymbol{x}$, according to any index in $\mathcal{B}_\mu$, if and only if the mean-normalised distribution of positive absolute income gaps in $\boldsymbol{y}$ general-Lorenz dominates $\boldsymbol{x}$'s *above and below the median*. When $m_{\boldsymbol{y}} = m_{\boldsymbol{x}}$ and $\mu_{\boldsymbol{y}} = \mu_{\boldsymbol{x}}$, the two conditions boil down to generalised-Lorenz dominance of $\boldsymbol{y}$ over $\boldsymbol{x}$ above the median coupled with inverse-generalised-Lorenz dominance of $\boldsymbol{x}$ over $\boldsymbol{y}$ below the median.

3.2.1 Asymmetric indices

We can identify the subclass of mean-normalised rank-independent indices, $\mathcal{B}_{\mu L} \subset \mathcal{B}_\mu$, fulfilling μ-type bottom asymmetry by imposing $\theta'_L(g) \geqslant \theta'_H(g)$ and $\theta''_L(g) \leqslant \theta''_H(g)$. Theorem 10 provides its robustness criteria:

Theorem 10. $\Delta\Psi_\mu(\boldsymbol{y},\boldsymbol{x}) \geqslant 0\ \forall \Psi_\mu \in \mathcal{B}_{\mu L}$ *if and only if* $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) \leqslant 0$ *and* $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) + \Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0\ \forall g \in [0,g^+]$.

Proof. See appendix A. ■

Comparing the conditions of theorem 10 with those of theorem 9, clearly the reduction in stringency comes about in the replacement of the generalised-Lorenz comparison above the median with the sum requirements $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) + \Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0\ \forall g \in [0,g^+]$. As in many other sequential dominance procedures, fulfillment of the first condition, i.e. $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) \leqslant 0\ \forall g \in [0,g^+]$, facilitates fulfillment of the second one, namely

$\Delta F^{(2)}(g;\boldsymbol{g}_L^*) + \Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0\ \forall g \in [0, g^+]$. By contrast, in the case of theorem 9, fulfillment of the generalised-Lorenz comparison below the median will not help to meet the respective second condition, i.e. the generalised-Lorenz comparison above the median.

Likewise, we can identify the subclass of mean-normalised rank-independent indices, $\mathcal{B}_{\mu H} \subset \mathcal{B}_\mu$, fulfilling μ-type top asymmetry by imposing $\theta'_L(g) \leqslant \theta'_H(g)$ and $\theta''_L(g) \geqslant \theta''_H(g)$. Then theorem 11 ensues:

Theorem 11. $\Delta\Psi_\mu(\boldsymbol{y},\boldsymbol{x}) \geqslant 0\ \forall \Psi_\mu \in \mathcal{B}_{\mu H}$ *if and only if* $\Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0$ *and* $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) + \Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0\ \forall g \in [0, g^+]$.

Proof. Similar to the proof of theorem 10. Left to the reader or available upon request. ■

Note the similarity between theorems 10 and 11. Of course a key difference is in the generalised-Lorenz comparisons that are left in or replaced by the sequential condition $\Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0$ and $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) + \Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0\ \forall g \in [0, g^+]$.

3.2.2 Symmetric indices

Finally we define and identify the subclass of mean-normalised rank-independent indices, $\mathcal{B}_{\mu S} \subset \mathcal{B}_\mu$, fulfilling μ-type symmetry with the imposition of $\theta_L(g) = \theta_H(g)\ \forall g \in [0, g^+]$. Theorem 12 provides the robustness conditions for this subclass:

Theorem 12. $\Delta\Psi_\mu(\boldsymbol{y},\boldsymbol{x}) \geqslant 0\ \forall \Psi_\mu \in \mathcal{B}_{\mu S}$ *if and only if* $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) + \Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0\ \forall g \in [0, g^+]$.

Proof. See appendix A. ■

Remarkably, the single dominance condition for subclass $\mathcal{B}_{\mu S}$ is less restrictive than the conditions of both $\mathcal{B}_{\mu L}$ and $\mathcal{B}_{\mu H}$, since the latter feature $\mathcal{B}_{\mu S}$'s condition together with further requirements. Hence, at least in this particular comparison of subclasses, the partial ordering generated by $\mathcal{B}_{\mu S}$ is the least incomplete among the studies classes of mean-normalised rank-independent bipolarisation indices based on absolute income gaps from the median.

4 Comparisons with existing dominance criteria

Some of the proposed dominance conditions coincide with proposals for robust ranking criteria from the literature. For instance, the conditions of theorem 1, which apply to general rank-dependent relative bipolarisation indices based on normalised gaps, coincide with the Foster-Wolfson curves (Foster and Wolfson, 2010) if the means are identical. Meanwhile with identical means and medians the conditions boil down to the robustness results of Bossert and Schworm (2008). Meanwhile, when the gaps are normalised by the mean, the conditions of theorem 4, which apply to symmetric rank-dependent indices, are identical to those related to the relative bipolarisation Lorenz curve (Yalonetzky, 2014).

The sequential dominance conditions for rank-dependent bipolarisation indices in theorems 2 and 3, respectively for bottom-asymmetric and top-asymmetric indices, stem from a combinations of the conditions laid out in theorems 1 and 4. Hence they do not correlate directly to other proposals in the literature, even under some comparison constraints (e.g. equal medians).

Theorem 5 provides robustness conditions for rank-independent and median-normalised relative bipolarisation indices. These conditions become the dual of those proposed by Bossert and Schworm (2008) whenever medians are identical. Meanwhile, when symmetry is imposed on this class, the ensuing conditions (theorem 8) will not generally be the dual counterpart of those related to the relative-bipolarisation Lorenz curve, unless the two distributions are symmetric themselves. Meanwhile the asymmetry-related sequential dominance conditions (stemming from theorems 6 and 7) will be, again, combinations of the conditions from the general class and the symmetric subclass of median-normalised, rank-independent indices. Therefore they are unlikely to have direct matches in the literature.

Finally, theorem 9 contains dominance conditions for mean-normalised rank-independent relative bipolarisation indices based on absolute income gaps from the median. When the means are equal, the pair of conditions matches the dual of the Foster-Wolfson curve. If, on top of that, the medians are equal, then the dual of the Bossert and Schworm (2008) condition arises. The robustness conditions for these indices' symmetric subclass are provided by theorem 12 and constitute the dual of the relative-bipolarisation Lorenz curve (Yalonetzky, 2014). As with previous classes and subclasses, the sequential dominance conditions for bottom- and top-asymmetric indices (stemming from theorems 10 and 11) combine conditions from the general class and the symmetric class of bipolarisation indices described above. Therefore they are unprecedented in the bipolarisation literature.

5 Empirical illustration

5.1 Data

The results presented in sections 2 and 3 are now illustrated using data from Georgia, the Caucasian nation, for the period 2009–2020. Household consumption series are available on a yearly basis from 2009 to 2016 with the Integrated Household Surveys and, since 2017, with the Household Incomes and Expenditures Surveys. The surveys cover all regions except the autonomous republic of Abkhazia and districts corresponding to South Ossetia. The number of surveyed households varies from 10,858 to 22,304. Monthly consumption series are based on cash and non-cash expenditures. An equivalence scale is used to ease the comparison of well-being levels between households of different sizes. Each individual in the survey is consequently given the value of their household's 'consumption per consumption unit'. The number of consumption units in a given household is its size to the 0.8 power.

Table 1 provides some summary statistics for the 12 compared consumption series. No-

Table 1: Description of consumption series for Georgia, 2009–2020.

Period	$\min\{\boldsymbol{y}\}$	$m_{\boldsymbol{y}}$	$\mu_{\boldsymbol{y}}$	$\max\{\boldsymbol{y}\}$	$\frac{\min\{\boldsymbol{y}\}}{\mu_{\boldsymbol{y}}}$	$\frac{m_{\boldsymbol{y}}}{\mu_{\boldsymbol{y}}}$	$\frac{\max\{\boldsymbol{y}\}}{\mu_{\boldsymbol{y}}}$	$\frac{\min\{\boldsymbol{y}\}}{m_{\boldsymbol{y}}}$	$\frac{\max\{\boldsymbol{y}\}}{m_{\boldsymbol{y}}}$
2009	0	152	195	16933	0	0.782	86.9	0	111
2010	0	163	213	29652	0	0.768	139	0	181
2011	0	184	231	29087	0	0.796	126	0	158
2012	6.42	196	248	5371	0.0259	0.79	21.7	0.0328	27.4
2013	3.33	221	277	5420	0.012	0.797	19.5	0.0151	24.5
2014	8.95	237	297	9004	0.0302	0.799	30.3	0.0377	37.9
2015	8.59	241	303	4977	0.0284	0.798	16.5	0.0356	20.6
2016	10.3	247	313	7860	0.033	0.789	25.1	0.0418	31.8
2017	2	268	334	10970	0.00599	0.801	32.9	0.00747	41
2018	9.09	270	333	10473	0.0273	0.81	31.5	0.0337	38.8
2019	13.9	289	353	19537	0.0393	0.819	55.4	0.048	67.7
2020	1.48	287	337	11958	0.0044	0.85	35.5	0.00518	41.7

Note: Amounts are in current Georgian Lari.

tably the median-to-mean ratio has increased over the years, which provides a valuable justification for contrasting bipolarizations orderings based on mean-normalized indices against those obtained for median-normalized indices. Moreover, a value lower than unity for this ratio is typical of positively skewed distributions, a distinctive feature of observed income and consumption distributions. The comparison of the ratios of $\min\{\boldsymbol{y}\}$ and $\max\{\boldsymbol{y}\}$ with respect to either the mean or the median is also indicative of this negative skewness. As indicated earlier, this means that, when focusing on relative and rank-independent bipolarization indices that comply with either μ-B-ASYM or m-B-ASYM, it will not be possible for distributional changes occurring for large gaps in the upper part of the distribution to be compensated by opposite changes in the lower part of the distribution.

5.2 Results

Table 2: Relative inequality and bipolarization orderings (rank-dependent indices) in Georgia, 2009–2020.

		Mean norm.				Median norm.			
period	Lorenz	NC	B-ASYM	T-ASYM	SYM	NC	B-ASYM	T-ASYM	SYM
2009–2010	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\preccurlyeq$	$\emptyset$	$\preccurlyeq$
2010–2011	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2011–2012	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2012–2013	$\succcurlyeq$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2013–2014	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2014–2015	$\succcurlyeq$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2015–2016	$\preccurlyeq$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2016–2017	$\emptyset$	$\emptyset$	$\preccurlyeq$	$\emptyset$	$\preccurlyeq$	$\emptyset$	$\preccurlyeq$	$\emptyset$	$\preccurlyeq$
2017–2018	$\succcurlyeq$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2018–2019	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\succcurlyeq$	$\emptyset$	$\emptyset$	$\emptyset$	$\succcurlyeq$
2019–2020	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\succcurlyeq$	$\succcurlyeq$
2009–2020	$\succcurlyeq$	$\emptyset$	$\emptyset$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$

Note: $\succcurlyeq$ ($\preccurlyeq$) indicates a dominance relationship of the final (initial) distribution over the initial (final) distribution; $\emptyset$ denotes an ambiguous ordering.

We first consider inequality orderings using the traditional Lorenz curve. Results for

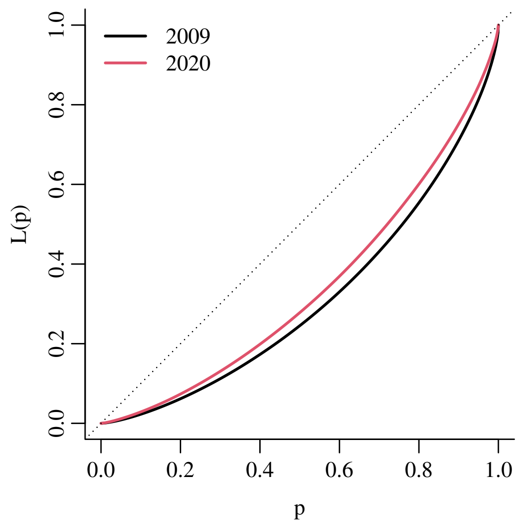


Figure 3: Lorenz curves for Georgia, 2009–2020.

year-to-year changes as well as for the whole period are given in table 2. $A \succcurlyeq B$ means that B is robustly less unequal (resp. bipolarised) than A. Since the time span is relatively short for year-to-year comparisons, it is not surprising that Lorenz dominance relationships are rather rare here. Nevertheless, among the eleven possible comparisons, we obtain four robust comparisons, with one increase (2015–2016) and three decreases in inequality (2012–2013, 2014–2015, and 2017–2018). Over the whole period, we also observe a robust decline in inequality as shown by figure 3.

Table 3: Bipolarization orderings (rank-independent indices) in Georgia, 2009–2020.

	Mean norm.				Median norm.			
period	NC	B-ASYM	T-ASYM	SYM	NC	B-ASYM	T-ASYM	SYM
2009-2010	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\preccurlyeq$	$\emptyset$	$\preccurlyeq$
2010-2011	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2011-2012	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2012-2013	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2013-2014	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2014-2015	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2015-2016	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2016-2017	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2017-2018	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2018-2019	$\emptyset$	$\succcurlyeq$	$\emptyset$	$\succcurlyeq$	$\emptyset$	$\succcurlyeq$	$\emptyset$	$\succcurlyeq$
2019-2020	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$
2009-2020	$\emptyset$	$\emptyset$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$	$\succcurlyeq$

Note: $\succcurlyeq$ ($\preccurlyeq$) indicates a dominance relationship of the final (initial) distribution over the initial (final) distribution; $\emptyset$ denotes an ambiguous ordering.

Our results regarding relative bipolarization orderings are given in table 2 for rank-dependent indices and in table 3 for rank-independent indices. In the case of the former, robust orderings are observed at best for 4 out of 11 year-to-year comparisons, whereas robust comparisons are only obtained for 3 year-to-year comparisons with rank-independent indices. Although this cannot be deemed a result with a universal reach, it is striking to note that observed robust bipolarization orderings are not associated with robust inequality orderings and vice versa.

Unconstrained robustness tests (NC columns in tables 2 and 3), that is without imposing either symmetry or any form of asymmetry, fail to deliver any robust year-to-year comparison in the case of Georgia, whichever the normalisation method. Increase in the ordering power can nevertheless be obtained if we consider indices that show more sensitivity for changes in the bottom or top part of the distribution when compared with the remaining half. It is worth noting that, in the case of Georgian consumption distributions, the addition of either μ-B-ASYM or μ-T-ASYM is of little consequence for partial orderings related to mean-normalized indices. More robust comparisons emerge for median-normalized indices, especially with m-B-ASYM and considering rank-independent bipolarization indices. Finally, restricting our attention to symmetric indices makes it possible to order at best four consecutive-year pairs of distributions with median-normalized and rank-dependent indices (2009–2010, 2016–2017, 2018–2019, and 2019–2020), but only one when focusing on mean-normalized rank-independent indices (2018–2019).

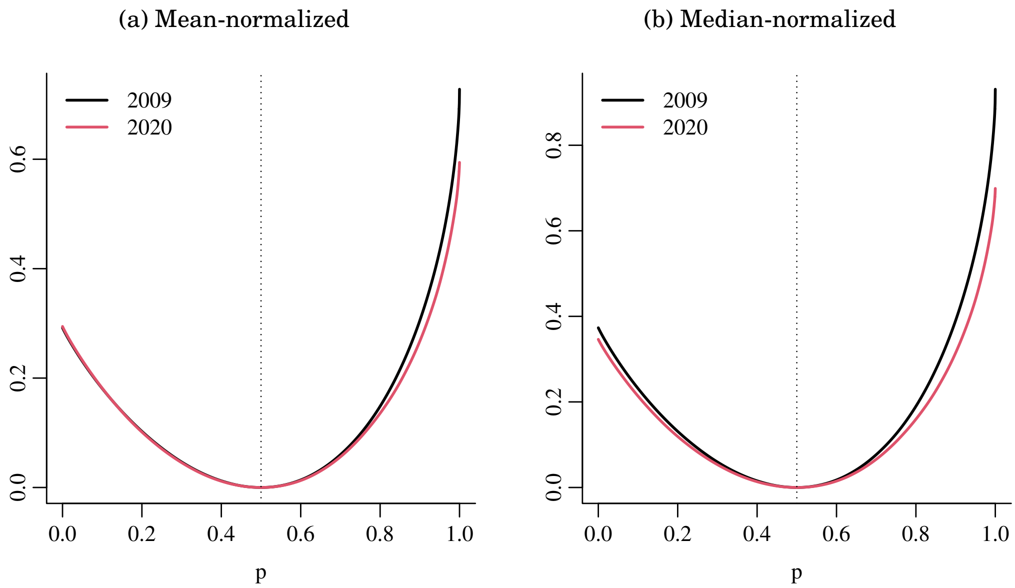


Figure 4: Polarization curves à la Foster-Wolfson for Georgia, 2009–2020.

For the whole period 2009–2020, interestingly, we observe both a robust decrease in relative inequalities and, under certain assumptions, a robust decrease in relative bipolarization. As shown in figure 4, even without the help of ASYM-type assumptions, we observe a robust decline in bipolarization when considering members of the class of rank-dependent median-normalized indices as the curve for 2020 is nowhere above the one for 2009. With mean-normalization, relative distance with respect to the median have unambiguously decreased in the top half of the distribution but, even if the two curves can hardly be distinguished for the bottom half, they indeed cross around the first decile of the population. Consequently, indices showing an extremely high sensitivity to mean-normalized gaps with respect to the median at the lowest quantile would lead us to conclude that relative bipolarization actually increased in Georgia between 2009 and 2020.

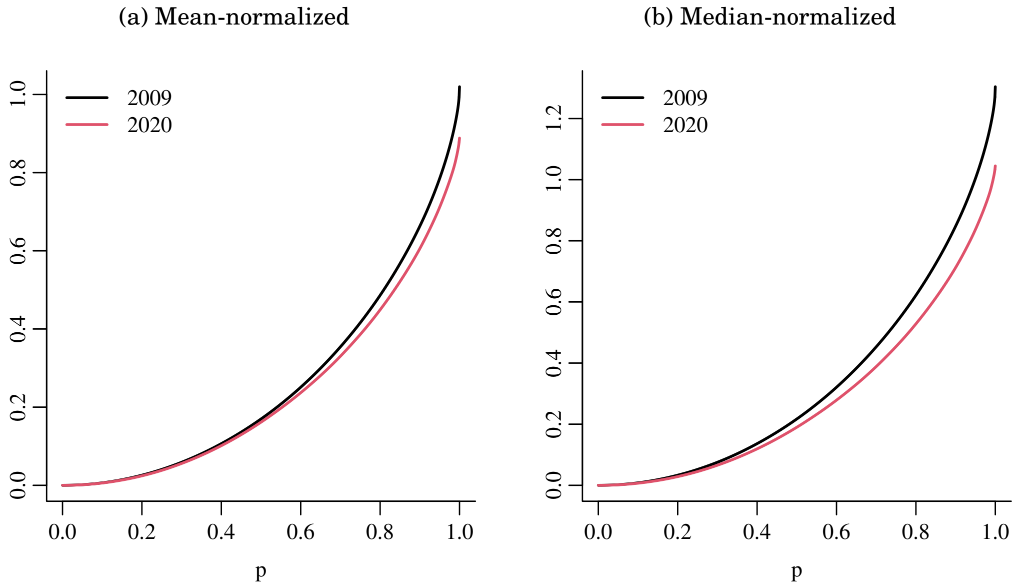


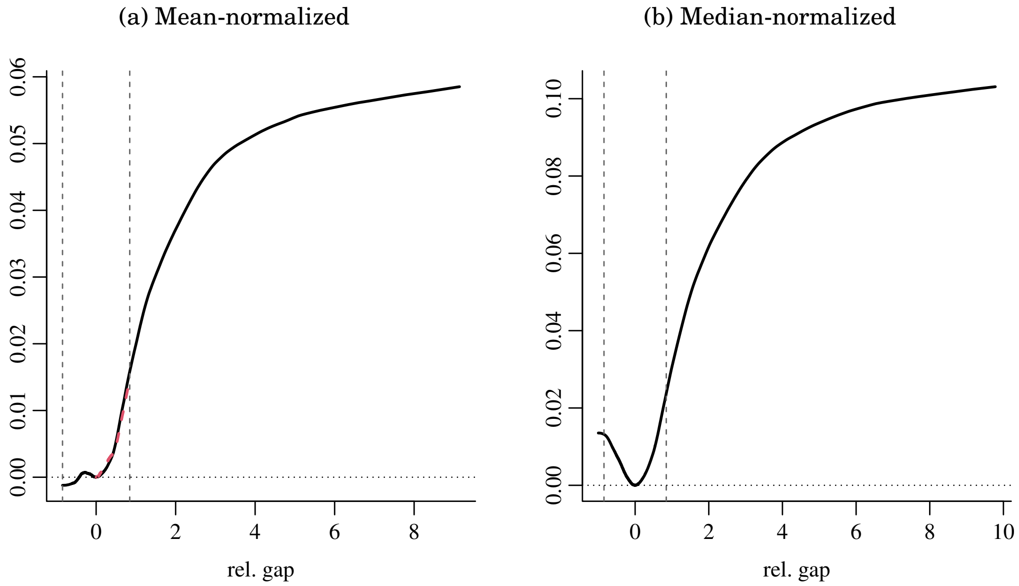
Figure 5: Aggregated curves for ASYM and SYM assumptions with rank-dependent indices.

Once μ-T-ASYM is imposed, a robust decrease in relative bipolarization holds (using the right part of figure 4a along with figure 5a).

Regarding rank-independent indices, the counterpart of figure 4 is of little help as the curves can hardly be distinguished. Figure 6 consequently represents the difference in $F^{(2)}$ or $\bar{F}^{(2)}$ for different values of the relative gaps (with negative values referring to consumption levels below the median). In the case of mean-normalized gaps, this difference is everywhere non-negative, except for gaps exceeding 42% of mean income in the bottom half of the distribution, hence indicating the possibility of having indices that show an increase in relative bipolarization. Once μ-T-ASYM is imposed, hence disregarding the curve for negative gaps and considering the red dashed curve instead, a robust decline in relative bipolarization can be ascertained. This additional assumption is not required in the case of median-normalized indices as shown by figure 6b, so that we can conclude that all members from $\mathcal{B}_m$ would stress a decrease in relative bipolarization in Georgia between 2009 and 2020.

6 Conclusion

Bipolarisation is not just a dispersion concept distinct from inequality. It is also internally multi-faceted. Even focusing on relative bipolarisation measurement satisfying scale invariance, we noted that indices may differ on an array of traits, including the distributional standard used for normalisation and, chiefly, the treatment of income gaps above and below the median. We showed that the latter is not only ethically meaningful intrinsically but it can also be used to gauge clustering behaviour above and below the median differentially. Combining both traits, namely normalisation choices and different notions of symmetry and asymmetry in the treatment of gaps, we identified a battery of partial



Note: the area between the two vertical dashed lines indicate the interval where compensation is likely to occur.

Figure 6: Difference in polarization curves for Georgia, 2009–2020.

orderings and their related stochastic dominance conditions. Moreover, the exercise was helpful in order to catalogue existing dominance criteria in the literature and relate it to specific subclasses of indices complying with different combinations of the aforementioned traits and properties. The lack of match between existing dominance criteria and the sequential dominance criteria for asymmetric bipolarisation indices bears testimony to the latter's novelty.

The illustration with consumption data from Georgia over the second decade of the 21st Century highlighted the empirical relevance of the identified partial orderings as, for instance, imposing the top-symmetry property brought about a picture of robust decline in bipolarisation between 2009 and 2020, highlighting the role of robust decrease in income gaps in the top half of Georgia's consumption distribution.

We divided relative bipolarisation indices in terms of their dependence on rank functions and defined broad classes of rank-dependent and rank-independent indices, in order to facilitate the derivation of dominance conditions. However, this should not be an impediment to making general robustness statements regarding broader classes of indices, which include both rank-dependent and rank-independent subclasses, since the dominance conditions pertaining to specific subclasses can always be combined and tested jointly. Furthermore, in many situations different subclasses may share the same robustness conditions and, hence, the same partial orderings. However in order to provide a tidier and tighter framework of classes of relative bipolarisation indices and their associated partial orderings, future work should seek to axiomatically characterise these classes of indices, going beyond the current characterisation by Bossert and Schworm (2008) in order to incorporate the symmetry and asymmetry axioms proposed in this paper. Then, from that starting point, a more parsimonious collection of dominance conditions can be derived.

Appendices

A Proof of dominance conditions

For the sake of simplicity, we assume that the weighing functions $w_L(p)$ and $w_H(p)$ have second-order derivatives all along the interval $[0,1]$. The same assumption also applies for $\psi_L(z)$, $\psi_H(z)$ and $\gamma(v)$ over the corresponding intervals. These assumptions can easily be slackened if we assume first-order derivatives are piecewise smooth over their domain of definition.

A.1 Theorems 1 to 4

Integrating (5) by parts, we obtain:

$$\begin{aligned}
\Psi^r(\boldsymbol{y}) &= \left[w_L(p)\int_0^p g_L(t)\,dt\right]_0^{0.5} - \int_0^{0.5} w_L'(p)\int_0^p g_L(t)\,dt\,dp + \big[w_H(p)G_H(p)\big]_{0.5}^1 \\
&\quad - \int_{0.5}^1 w_H'(p)G_H(p)\,dp, \qquad (9)\\
&= \left[w_L(p)\int_0^p g_L(t)\,dt\right]_0^{0.5} - \int_0^{0.5} w_L'(p)\left(\int_0^{0.5} g_L(t)\,dt - \int_p^{0.5} g_L(t)\,dt\right)dp \\
&\quad + \big[w_H(p)G_H(p)\big]_{0.5}^1 - \int_{0.5}^1 w_H'(p)G_H(p)\,dp, \qquad (10)\\
&= w_L(0)G_L(0) + \int_0^{0.5} w_L'(p)G_L(p)\,dp + w_H(1)G_H(1) - \int_{0.5}^1 w_H'(p)G_H(p)\,dp, \qquad (11)
\end{aligned}$$

since $\int_0^0 g_L(p)\,dp = G_H(0.5) = 0$. As $w_L(0) \geqslant 0$, $w_H(1) \geqslant 0$, $w_L'(p) \geqslant 0$, $w_H'(p) \leqslant 0$ $\forall p \in [0,1]$ a sufficient condition for $\Delta\Psi^r(\boldsymbol{y},\boldsymbol{x}) \geqslant 0$ is $\Delta G_L(p) \geqslant 0$ $\forall p \in [0, 0.5]$ and $\Delta G_H(p) \geqslant 0$ $\forall p \in [0.5, 1]$. For necessity, note that $\Delta G_H(p)$ is unbounded as its two components are themselves unbounded. Therefore, if we let $\Delta G_H(p) \to -\infty$ with all weights lower than infinity, we would get $\Delta\Psi^r(\boldsymbol{y},\boldsymbol{x}) < 0$. Likewise, as $w_L(p)$ is also unbounded from above, we could have a situation where $w_L(p) \to 0$ with $\Delta G_L(p) < 0$ and $\Delta G_L(q) = 0$ for $p < q \leqslant 0.5$. In that case we would get again $\Delta\Psi^r(\boldsymbol{y},\boldsymbol{x}) < 0$.

Now, let us assume $w_L(p) \geqslant w_H(p)$ $\forall p \in [0,1]$. Equation (11) can then be rewritten as:

$$\begin{aligned}
\Psi^r(\boldsymbol{y}) &= w_L(0)G_L(0) + (1-1)w_H(1)G_L(0) - \int_0^{0.5} w_L'(p)G_L(p)\,dp \\
&\quad + (1-1)\int_{0.5}^1 w_H'(p)G_L(1-p)\,dp + w_H(1)G_H(1) - \int_{0.5}^1 w_H'(p)G_H(p)\,dp, \qquad (12)\\
&= \big(w_L(0) - w_H(1)\big)G_L(0) - \int_0^{0.5}\big(w_L'(p) - w_H'(1-p)\big)G_L(p)\,dp \\
&\quad + w_H(1)\big(G_L(0) + G_H(1)\big) - \int_{0.5}^1 w_H'(p)\big(G_L(1-p) + G_H(p)\big)\,dp. \qquad (13)
\end{aligned}$$

As m-B-ASYM and μ-B-ASYM mean $w_L(p) - w_H(1-p) \geqslant 0$ and $w_L'(p) - w_H'(1-p) \leqslant$

$0 \; \forall p \in [0, 0.5]$, a sufficient condition for $\Delta\Psi^r(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0$ is $\Delta G_L(p) \leqslant 0 \; \forall p \in [0, 0.5]$ and $\Delta\big(G_L(1-p) + G_H(p)\big) \geqslant 0 \; \forall p \in [0.5, 1]$. In the case of either m-SYM or μ-SYM, the first two terms in (13) vanish, so that a sufficient condition for $\Delta\Psi^r(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0$ becomes $\Delta\big(G_L(1 - p) + G_H(p)\big) \geqslant 0 \; \forall p \in [0.5, 1]$. For necessity we can deploy the same reasoning as in the previous proof.

A.2 Theorems 5 to 8

Starting from (7) and integrating by parts, we obtain:

$$\begin{aligned}
\Psi_m(\boldsymbol{y}) &= [\psi_L(z) F(z; \boldsymbol{z}_L^*)]_0^{z^+} - \int_0^{z^+} \psi_L'(z) F(z; \boldsymbol{z}_L^*)\, dz \\
&\quad + [\psi_H(z) F(z; \boldsymbol{z}_H^*)]_0^{z^+} - \int_0^{z^+} \psi_H'(z) F(z; \boldsymbol{z}_H^*)\, dz, && (14) \\
&= \psi_L(z^+) - \int_0^{z^+} \psi_L'(z) F(z; \boldsymbol{z}_L^*)\, dz + \psi_H(z^+) - \int_0^{z^+} \psi_H'(z) F(z; \boldsymbol{z}_H^*)\, dz && (15) \\
&= \psi_L(z^+) - \int_0^{z^+} \psi_L'(z)\big(1 - \bar{F}(z; \boldsymbol{z}_L^*)\big)\, dz + \psi_H(z^+) - \int_0^{z^+} \psi_H'(z) F(z; \boldsymbol{z}_H^*)\, dz && (16) \\
&= \psi_L(z^+) - [\psi_L(z)]_0^{z^+} - \int_0^{z^+} \psi_L'(z)\big(- \bar{F}(z; \boldsymbol{z}_L^*)\big)\, dz + \psi_H(z^+) - \int_0^{z^+} \psi_H'(z) F(z; \boldsymbol{z}_H^*)\, dz && (17) \\
&= \psi_L(0) - \int_0^{z^+} \psi_L'(z)\big(- \bar{F}(z; \boldsymbol{z}_L^*)\big)\, dz + \psi_H(z^+) - \int_0^{z^+} \psi_H'(z) F(z; \boldsymbol{z}_H^*)\, dz && (18) \\
&= \psi_L(0) - \Big[\psi_L'(z) \bar{F}^{(2)}(z; \boldsymbol{z}_L^*)\Big]_0^{z^+} + \int_0^{z^+} \psi_L''(z) \bar{F}^{(2)}(z; \boldsymbol{z}_L^*)\, dz \\
&\quad + \psi_H(z^+) - \Big[\psi_H'(z) F^{(2)}(z; \boldsymbol{z}_H^*)\Big]_0^{z^+} + \int_0^{z^+} \psi_H''(z) F^{(2)}(z; \boldsymbol{z}_H^*)\, dz && (19) \\
&= \psi_L(0) + \psi_L'(0) \bar{F}^{(2)}(z^-; \boldsymbol{z}_L^*) + \int_0^{z^+} \psi_L''(z) \bar{F}^{(2)}(z; \boldsymbol{z}_L^*)\, dz \\
&\quad + \psi_H(z^+) - \psi_H'(z^+) F^{(2)}(z^+; \boldsymbol{z}_H^*) + \int_0^{z^+} \psi_H''(z) F^{(2)}(z; \boldsymbol{z}_H^*)\, dz. && (20)
\end{aligned}$$

Considering the difference $\Delta\Psi(\boldsymbol{y})$ in bipolarization between two income distributions, we obtain from (20):

$$\begin{aligned}
\Delta\Psi_m(\boldsymbol{y}) &= \psi_L'(0) \Delta\bar{F}^{(2)}(0; \boldsymbol{z}_L^*) + \int_0^{z^+} \psi_L''(z) \Delta\bar{F}^{(2)}(z; \boldsymbol{z}_L^*)\, dz \\
&\quad - \psi_H'(z^+) \Delta F^{(2)}(z^+; \boldsymbol{z}_H^*) + \int_0^{z^+} \psi_H''(z) \Delta F^{(2)}(z; \boldsymbol{z}_H^*)\, dz, && (21) \\
&= \psi_L'(0) \Delta\bar{F}^{(2)}(0; \boldsymbol{z}_L^*) + \int_0^{1} \psi_L''(z) \Delta\bar{F}^{(2)}(z; \boldsymbol{z}_L^*)\, dz \\
&\quad - \psi_H'(z^+) \Delta F^{(2)}(z^+; \boldsymbol{z}_H^*) + \int_1^{z^+} \psi_H''(z) \Delta F^{(2)}(z; \boldsymbol{z}_H^*)\, dz, && (22)
\end{aligned}$$

since, by construction, $\Delta\bar{F}^{(2)}(z; \boldsymbol{z}_L^*) = 0 \; \forall z \in [1, z^+]$ and $\Delta F^{(2)}(z; \boldsymbol{z}_H^*) = 0 \; \forall z \in [0, 1]$. With

$\psi'_L(z) \leqslant 0$, $\psi''_L(z) \leqslant 0$,$\psi'_H(z) \geqslant 0$, and $\psi''_H(z) \leqslant 0 \; \forall z \in [0, z^+]$, it follows that $\Delta\bar{F}^{(2)}(z; \boldsymbol{z}^*_L) \leqslant 0$ $\forall z \in [0, 1]$ and $\Delta F^{(2)}(z; \boldsymbol{z}^*_H) \leqslant 0 \; \forall z \in [1, z^+]$ are sufficient conditions for $\Delta\Psi(\boldsymbol{y}) \geqslant 0$.

For necessity, let us consider the case where *i)* ψ''_L is equal to zero at each point within the interval $[0, z^+]$ except $a < 1$ where $\psi''_L(a) < 0$ and *ii)* ψ''_H is equal to zero at each point within the interval $[0, z^+]$. It then can easily be seen that for $\Delta\Psi \geqslant 0$ given the restrictions on the sign of ψ''_L it is necessary to have $\Delta\bar{F}^{(2)}(a; \boldsymbol{z}^*_L) \leqslant 0$. In the same way, if we assume *i)* ψ''_L is equal to zero at each point within the interval $[0, z^+]$ and *ii)* ψ''_H is equal to zero at each point within the interval $[0, z^+]$ except $b > 1$ where $\psi''_L(b) < 0$, we observe that a necessary condition for $\Delta\Psi \geqslant 0$ is $\Delta F^{(2)}(b; \boldsymbol{z}^*_H) \leqslant 0$.

Now, we consider members from $\mathcal{B}_{mL}$. Assuming $2 \leqslant z^+$, (22) can be written as:

$$\begin{aligned}
\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) &= \psi'_L(0)\Delta\bar{F}^{(2)}(0; \boldsymbol{z}^*_L) + \int_1^2 \psi''_L(2-z)\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L)\, dz \\
&\quad - \psi'_H(z^+)\Delta F^{(2)}(z^+; \boldsymbol{z}^*_H) + \int_1^{z^+} \psi''_H(z)\Delta F^{(2)}(z; \boldsymbol{z}^*_H)\, dz, \qquad (23)\\
&= \psi'_L(0)\Delta\bar{F}^{(2)}(0; \boldsymbol{z}^*_L) + \int_1^2 \left(\psi''_L(2-z) + (1-1)\psi''_H(z)\right) \Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L)\, dz \\
&\quad - \psi'_H(z^+)\Delta F^{(2)}(z^+; \boldsymbol{z}^*_H) + \int_1^{z^+} \psi''_H(z)\Delta F^{(2)}(z; \boldsymbol{z}^*_H)\, dz, \qquad (24)\\
&= \psi'_L(0)\Delta\bar{F}^{(2)}(0; \boldsymbol{z}^*_L) + \int_1^2 \left(\psi''_L(2-z) - \psi''_H(z)\right) \Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L)\, dz \\
&\quad - \psi'_H(z^+)\Delta F^{(2)}(z^+; \boldsymbol{z}^*_H) + \int_1^2 \psi''_H(z) \left(\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L) + \Delta F^{(2)}(z; \boldsymbol{z}^*_H)\right)\, dz \\
&\quad + \int_2^{z^+} \psi''_H(z)\Delta F^{(2)}(z; \boldsymbol{z}^*_H)\, dz. \qquad (25)
\end{aligned}$$

Since m-B-ASYM notably means $\psi''_L(z) \leqslant \psi''_H(2-z)$, it follows from inspection that sufficient conditions for $\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) \geqslant 0$ are $\Delta\bar{F}^{(2)}(z; \boldsymbol{z}^*_L) \leqslant 0 \; \forall z \in [0, 1]$, $\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L) + \Delta F^{(2)}(z; \boldsymbol{z}^*_H) \leqslant 0 \; \forall z \in [1, 2]$, and $\Delta F^{(2)}(z; \boldsymbol{z}^*_H) \leqslant 0 \; \forall z \in [2, z^+]$. Necessity can be ascertained using similar reasoning as deployed for proofs above.

With m-SYM, $\psi''_L(2-z) = \psi''_H(z) \; \forall z$ so that the second element in (25) vanishes. Moreover:

$$\begin{aligned}
\Delta\Psi_m(\boldsymbol{y}, \boldsymbol{x}) &= \psi'_L(0)\Delta\bar{F}^{(2)}(0; \boldsymbol{z}^*_L) + (1-1)\psi'_L(0)\Delta F^{(2)}(2; \boldsymbol{z}^*_H) \\
&\quad - \psi'_H(z^+)\Delta F^{(2)}(z^+; \boldsymbol{z}^*_H) + \int_1^2 \psi''_H(z) \left(\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L) + \Delta F^{(2)}(z; \boldsymbol{z}^*_H)\right)\, dz \\
&\quad + \int_2^{z^+} \psi''_H(z)\Delta F^{(2)}(z; \boldsymbol{z}^*_H)\, dz, \qquad (26)\\
&= \psi'_L(0) \left(\Delta\bar{F}^{(2)}(0; \boldsymbol{z}^*_L) + \Delta F^{(2)}(2; \boldsymbol{z}^*_H)\right) - \psi'_H(0)\Delta F^{(2)}(2; \boldsymbol{z}^*_H) \\
&\quad - \psi'_H(z^+)\Delta F^{(2)}(z^+; \boldsymbol{z}^*_H) + \int_1^2 \psi''_H(z) \left(\Delta\bar{F}^{(2)}(2-z; \boldsymbol{z}^*_L) + \Delta F^{(2)}(z; \boldsymbol{z}^*_H)\right)\, dz \\
&\quad + \int_2^{z^+} \psi''_H(z)\Delta F^{(2)}(z; \boldsymbol{z}^*_H)\, dz, \qquad (27)
\end{aligned}$$

Sufficient conditions for $\Delta\Psi_m(\boldsymbol{y},\boldsymbol{x}) \geqslant 0$ are consequently $\Delta\bar{F}^{(2)}(2-z;\boldsymbol{z}_L^*)+\Delta F^{(2)}(z;\boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [1,2]$, and $\Delta F^{(2)}(z;\boldsymbol{z}_H^*) \leqslant 0\ \forall z \in [2,z^+]$. Necessity can be ascertained using similar reasoning as deployed in the above proofs.

A.3 Theorems 9 to 12

Integrating (9) twice by parts yields:

$$\begin{aligned}
\Psi_m u(\boldsymbol{y}) &= \left[\theta_L(g)F(g;\boldsymbol{g}_L^*)\right]_0^{g^+} - \int_0^{g^+} \theta_L'(g)F(g;\boldsymbol{g}_L^*)\,dg \\
&\quad + \left[\theta_H(g)F(g;\boldsymbol{g}_H^*)\right]_0^{g^+} - \int_0^{g^+} \theta_H'(g)F(g;\boldsymbol{g}_H^*)\,dg, &(28)\\
&= \theta_L(g^+) - \int_0^{g^+} \theta_L'(g)F(g;\boldsymbol{g}_L^*)\,dg + \theta_H(g^+) - \int_0^{g^+} \theta_H'(g)F(g;\boldsymbol{g}_H^*)\,dg &(29)\\
&= \theta_L(g^+) + \theta_H(g^+) - \left[\theta_L'(g)F^{(2)}(g;\boldsymbol{g}_L^*)\right]_0^{g^+} + \int_0^{g^+} \theta_L''(g)F^{(2)}(g;\boldsymbol{g}_L^*)\,dg \\
&\quad - \left[\theta_H'(g)F^{(2)}(g;\boldsymbol{g}_H^*)\right]_0^{g^+} + \int_0^{g^+} \theta_H''(g)F^{(2)}(g;\boldsymbol{g}_H^*)\,dg, &(30)\\
&= \theta_L(g^+) + \theta_H(g^+) - \theta_L'(g^+)F^{(2)}(g^+;\boldsymbol{g}_L^*) + \int_0^{g^+} \theta_L''(g)F^{(2)}(g;\boldsymbol{g}_L^*)\,dg \\
&\quad - \theta_H'(g^+)F^{(2)}(g^+;\boldsymbol{g}_H^*) + \int_0^{g^+} \theta_H''(g)F^{(2)}(g;\boldsymbol{g}_H^*)\,dg. &(31)
\end{aligned}$$

Comparing the change in bipolarization when moving from one distribution to another one, we have:

$$\begin{aligned}
\Delta\Psi_m u(\boldsymbol{y},\boldsymbol{x}) &= -\theta_L'(g^+)\Delta F^{(2)}(g^+;\boldsymbol{g}_L^*) + \int_0^{g^+} \theta_L''(g)\Delta F^{(2)}(g;\boldsymbol{g}_L^*)\,dg \\
&\quad - \theta_H'(g^+)\Delta F^{(2)}(g^+;\boldsymbol{g}_H^*) + \int_0^{g^+} \theta_H''(g)\Delta F^{(2)}(g;\boldsymbol{g}_H^*)\,dg. &(32)
\end{aligned}$$

Since $\theta_L'(g) \geqslant 0$, $\theta_H'(g) \geqslant 0$, $\theta_L''(g) \leqslant 0$, and $\theta_H''(g) \leqslant 0$, we can conclude that $\Delta F^{(2)}(g;\boldsymbol{g}_L^*) \leqslant 0$ and $\Delta F^{(2)}(g;\boldsymbol{g}_H^*) \leqslant 0$ are sufficient conditions for $\Delta\Psi_m u \geqslant 0$.

For necessity, let consider the case where *i)* θ_L'' is equal to zero at each point except at $a \in [0,g^+]$ where $\theta_L''(a) < 0$ and *ii)* θ_H'' is equal to zero at each point within the interval $[0,g^+]$. It then can easily be seen that for $\Delta\Psi_m u(\boldsymbol{y},\boldsymbol{x}) \leqslant 0$ given the restrictions on the sign of θ_L'' it is necessary to have $\Delta\bar{F}^{(2)}(a;\boldsymbol{g}_L^*) \leqslant 0$. In the same way, if we assume *i)* θ_L'' is equal to zero at each point within the interval $[0,g^+]$ and *ii)* θ_H'' is equal to zero at each point except at $b \in [0,g^+]$ where $\theta_L''(b) < 0$, we observe that a necessary condition for $\Delta\Psi_m u(\boldsymbol{y},\boldsymbol{x}) \leqslant 0$ is $\Delta F^{(2)}(b;\boldsymbol{g}_H^*) \leqslant 0$.

Considering μ-B-ASYM, (32) can be expressed as:

$$\Delta\Psi_m u(\boldsymbol{y},\boldsymbol{x}) = -\theta_L'(g^+)\Delta F^{(2)}(g^+;\boldsymbol{g}_L^*) + (1-1)\theta_H'(g^+)\Delta F^{(2)}(g^+;\boldsymbol{g}_L^*) - \theta_H'(g^+)\Delta F^{(2)}(g^+;\boldsymbol{g}_H^*)$$

$$+\int_0^{g^+}\left(\theta_L''(g)+(1-1)\theta_H''(g)\right)\Delta F^{(2)}(g;\boldsymbol{g}_L^*)\,dg+\int_0^{g^+}\theta_H''(g)\Delta F^{(2)}(g;\boldsymbol{g}_H^*)\,dg, \tag{33}$$

$$=-\left(\theta_L'(g^+)-\theta_H'(g^+)\right)\Delta F^{(2)}(g^+;\boldsymbol{g}_L^*)-\theta_H'(g^+)\left(\Delta F^{(2)}(g^+;\boldsymbol{g}_L^*)+\Delta F^{(2)}(g^+;\boldsymbol{g}_H^*)\right)$$
$$+\int_0^{g^+}\left(\theta_L''(g)-\theta_H''(g)\right)\Delta F^{(2)}(g;\boldsymbol{g}_L^*)\,dg+\int_0^{g^+}\theta_H''(g)\left(\Delta F^{(2)}(g;\boldsymbol{g}_L^*)+\Delta F^{(2)}(g;\boldsymbol{g}_H^*)\right)dg. \tag{34}$$

Since $\theta_L'(g)\geqslant\theta_H'(g)$ and $\theta_L''(g)\leqslant\theta_H''(g)$ are assumed for members from $\mathcal{B}_{\mu L}'$, sufficient conditions for $\Delta\Psi_m u\leqslant 0$ are $\Delta F^{(2)}(g;\boldsymbol{g}_L^*)$ and $\Delta F^{(2)}(g;\boldsymbol{g}_L^*)+\Delta F^{(2)}(g;\boldsymbol{g}_H^*)$ $\forall g\in[0,g^+]$. The reasoning behind the necessity part of the proof is very similar to that deployed in the proof of theorem 9.

Regarding μ-SYM, the restriction $\theta_L(g)=\theta_H(g)$ $\forall g\in[0,g^+]$ means that the first and third terms in (34) boils down, so that $\Delta F^{(2)}(g;\boldsymbol{g}_L^*)+\Delta F^{(2)}(g;\boldsymbol{g}_H^*)$ $\forall g\in[0,g^+]$ is a sufficient condition for $\Delta\Psi_m u\leqslant 0$. The reasoning behind the necessity part of the proof is very similar to that deployed in the proof of theorem 9.

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