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A rank-dependent multidimensional deprivation index (MDI) for binary data.†

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Abstract

This paper first shows how to apply the Sen-Shorrocks poverty index to the analysis of multidimensional deprivation, when only dichotomous variables are available to assess deprivation in various domains, the most common case in the literature. More precisely, it introduces a rank-dependent multidimensional poverty index, using a counting approach. The resulting multidimensional deprivation index, or *MDI* in short, has a nice graphical representation ("PUB curve") that turns out to be an extension of the so-called *TIP* curve of Jenkins and Lambert to the case of multiple deprivations. This graphical representation is very similar to the *SD* curve introduced by Lasso de la Vega (2010), but additionally emphasizes the third "I" of multidimensional deprivation: inequality. The *MDI* is sensitive to inequality and satisfies quite nice properties, but it cannot be broken down by population subgroups, when a traditional decomposition is used, and it does not have the property of dimensional breakdown, as the latter is usually defined in the literature. The paper proves, however, that there exists an alternative decomposition by population subgroups that can be applied to the *MDI;* it also derives a decomposition by deprivation domain, analogous to the breakdown of the Gini index by factor components. An empirical illustration based on deprivation data from four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua) shows the usefulness of the *MDI*.

Keywords: Multidimensional poverty analysis; Inequality; Gini index; Dominance.

JEL Codes: I3; I31; I32; D6; D63; O1; H1

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1. Introduction

To understand the threat posed by the problem of poverty, it is necessary to know the extent of interdependence between the various dimensions of poverty, its determinants, and the process through which it appears to deepen. In this context, an important question concerns the way poverty, and its changes, should be measured (Chakravarty, 2006). As noted by Thorbecke (2007, p. 4), before poverty can be measured, it has at least to be understood conceptually. In this regard, following the seminal contributions of Amartya Sen and his theoretical framework of "capabilities and functionings" (Sen, 1985, 1992, 1993, 2000), and earlier work on the measurement of multidimensional welfare and inequality (Kolm, 1977; Atkinson and Bourguignon, 1982; Maasoumi, 1986; Tsui, 1995; Maasoumi, 1999; Bourguignon, 1999), our conceptual understanding of poverty improved considerably. There is now quite a consensus on the multidimensional nature of poverty. As a result, since the pioneering works of Atkinson (2003), Bourguignon and Chakravarty (2003), and Tsui (2002), a number of approaches to analyzing and measuring multidimensional poverty and deprivation have been proposed in the literature (see, for example, Aaberge, Peluso, and Sigstad, 2019; Alkire and Foster, 2011; Bossert, Chakravarty, and D'Ambrosio, 2013; Datt, 2019; Dhongde, Li, Pattanaik, and Xu, 2016; Duclos, Sahn, and Younger, 2008; Kakwani and Silber, 2008; Lemmi and Betti, 2006, 2013; Pattanaik and Xu, 2018; Permanyer, 2014; Rippin, 2013, 2017).

Of particular interest are the works of Chakravarty, Mukherjee and Renade (1998), Tsui (2002), and Bourguignon and Chakravarty (2003) who have defined a poverty line for each dimension and then combined these different poverty thresholds and the domain-specific poverty gaps into a multidimensional poverty measure. Atkinson (2003) has also made an important contribution, firstly because his paper focused on the contrast between a social welfare approach and a counting approach to multidimensional poverty measurement, secondly because it provided a very thorough discussion of how to integrate into the analysis the interaction between the various dimensions of poverty.

Currently, the most popular methodology in the literature on multidimensional poverty analysis is the counting approach proposed by Alkire and Foster (2011), which is applied in the definition of the global multidimensional poverty index or global *MPI* (Alkire and Santos, 2010, 2014), the best known and most influential application of this method (Duclos and Tiberti, 2016; Pogge and Wisor, 2016). This methodology uses a "dual cutoff method" for the identification of the multidimensional poor (Alkire & Foster, 2011, p. 478), an essential and innovative feature (Datt, 2019), which includes the traditional union and intersection approaches as special cases (Atkinson, 2003). Alkire & Foster (2011, p. 479) introduced also a “class of multidimensional poverty measures (M_α)” for aggregating the information on the poor (Alkire & Foster, 2011, p. 479), which is an extension of the *FGT* monetary poverty measures (Foster, Greer, & Thorbecke, 1984). Alkire and Foster’s approach has, however, some shortcomings which may challenge, for example, the fulfillment of the overarching concern of the SDGs: “leaving no one behind” (Klasen and Fleurbaey, 2019, p. 1). These deficiencies have been mentioned, for example, in Aaberge and Brandolini (2015) and discussed in depth by Pattanaik and Xu (2018), as well as by Datt (2019), Duclos and Tiberti (2016), Espinoza-Delgado and Silber (2021), and Rippin (2017).

A different view of multidimensional deprivation measurement was adopted by Chakravarty and D’Ambrosio (2006), who took a counting approach and proposed a measure of social exclusion, while Yalonetzky (2014), as well as Silber and Yalonetzky (2013), have proposed a general formulation that includes, as special cases, the approaches of Alkire and Foster (2011), Chakravarty and D’Ambrosio (2006), Rippin (2010) and Bossert et al. (2013). Another interesting contribution is that of Aaberge and Peluso (2012) and Aaberge, Peluso, and Sigstad (2019) who assumed that the social poverty function is directly a function of the proportions of individuals with 1, 2,...*D* deprivations (see also an extension of this approach by Silber and Yalonetzky, 2013)

In the present paper, we also focus on discrete variables, in fact on dichotomous (binary) variables, the most common case in the literature. The key contribution of the paper is that it introduces a rank-dependent and an inequality sensitive multidimensional poverty index for multiple binary indicators, using a counting approach. The proposed index is an extension of the famous Sen-Shorrocks unidimensional poverty index (Shorrocks, 1995) to the measurement of multidimensional deprivation; we call this extension “*MDI*”: multidimensional deprivation index. The Sen-Shorrocks index has many useful properties that turn out to have important policy implications when applied to the multidimensional case. Although, in principle, the *MDI* cannot be broken down by population subgroups when a traditional decomposition is used, and it does not have the property of dimensional breakdown, as the latter is usually defined in the literature, we prove that there exists an alternative decomposition by population subgroups that can be applied to this index. We also derive a decomposition of the *MDI* by deprivation domain that is analogous to the breakdown of the Gini index by factor components.

Moreover, since the Sen-Shorrocks index can be interpreted graphically, we can compare the deprivation profiles of various countries or of different age groups and regions. Thus, we also extend the *TIP* curve introduced by Jenkins and Lambert (1997; 1998a; 1998b) to the multidimensional case. Note that the graphical representation we obtain is very similar to the *SD* curve introduced by Lasso de la Vega (2010), but presented in another context. We also prove that the *MDI* is related to a specific case of the Aaberge et al. (2019) deprivation measure. Finally, an empirical illustration focusing on Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua) shows the usefulness of the *MDI*.

The paper is organized as follows. In Section 2, we summarize some previous attempts to measure multidimensional poverty when only binary variables are available. Section 3 introduces what we call the multidimensional deprivation index (*MDI*), which is an extension of the approach of Shorrocks (1995) to the case of multidimensional deprivation with dichotomous variables. Section 4 presents the properties of the *MDI*. Section 5 provides an empirical illustration based on data from Central American countries, while Section 6 offers concluding remarks. An Online Appendix provides simple illustrations of the various properties of the *MDI* and of the similarity between the *MDI* and a specific case of the Aaberge et al. (2019) measure.

2. On previous attempts of measuring multidimensional poverty when only binary variables are available

2.1. The approach of Chakravarty and D'Ambrosio (2006)

These authors derived axiomatically a measure of social exclusion that can be also interpreted as a measure of multidimensional deprivation, as shown by Jayaraj and Subramanian (2010). Let P refer to the total number of deprivation dimensions, P_i to the number of domains in which individual i is deprived and n to the size of the population. The set of poor individuals will be defined as $\{i \mid P_i = P\}$ when an intersection approach is adopted, as $\{i \mid P_i \geq 1\}$ when taking a union approach and as $\{i \mid P_i \geq r\}$ when adopting the Alkire and Foster intermediate approach, r referring to the minimum number of domains in which an individual must be deprived to be considered as "overall poor". Following Chakravarty and D'Ambrosio (2006), when taking a union approach, one can then define an individual deprivation function d_i as $d_i = 0$ if individual i is not deprived in any dimension and as $d_i = \left(\frac{P_i}{P}\right)^{\alpha}$ when individual i is deprived in P_i dimensions, with $\alpha > 0$. The level of deprivation in the society as a whole will then be expressed as

$$D = \left(\frac{1}{n}\right) \sum_{i=1}^{n} d_i = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left(\frac{P_i}{P}\right)^{\alpha} = \sum_{j=1}^{K} H_j \left(\frac{j}{P}\right)^{\alpha} \quad (1)$$

where H_j is the proportion of individuals deprived in exactly j dimensions.

In the specific case where $\alpha = 1$, expression (1) will be written as

$$D = \sum_{j=1}^{K} H_j \left(\frac{j}{P}\right) \quad (2)$$

If we adopt the intermediate approach of Alkire and Foster (2011), expression (1) will be written as

$$D = \sum_{j \geq r}^{P} H_j \left(\frac{j}{P}\right) = \sum_{j \geq r}^{P} \left(\frac{n_j}{n}\right) \left(\frac{j}{P}\right) \quad (3)$$

where n_j refers to the number of individuals who have j deprivations.

But (3) may be also written as

$$D = \left(\frac{\sum_{j \geq r}^{P} n_j}{n}\right) \left(\frac{\sum_{j \geq r}^{P} j n_j}{\left(\sum_{j \geq r}^{P} n_j\right) K}\right) = H \times A \quad (4)$$

where H is the headcount ratio when adopting the intermediate approach of Alkire Foster with an overall threshold of r, while A is what Alkire and Foster (2011) call the average deprivation share across those classified as poor (deprived). Expression (4) refers in fact to what Alkire and Foster (2011, p. 479) called “the adjusted headcount ratio” ("M_0").

2.2. Additional approaches to multidimensional deprivation measurement with binary variables

There have been other attempts to measure multidimensional deprivation when only binary variables are available. As stressed by Dhongde et al. (2016), although in the literature on multidimensional poverty there are quite a few studies using discrete data (e.g., Alkire and

Foster, 2011; Bossert et al., 2013; Lasso de la Vega, 2010), relatively few stress the specific case of binary data. Fusco and Dickes (2006) used binary data but did not propose or derive an index; they used a Rasch (1960) model. Rippin (2010) introduced a multidimensional poverty index for the case of discrete data but did not specially focus on binary data. Finally Dhongde et al. (2016) made an interesting distinction between basic attributes and non-basic attributes, where each basic attribute has priority over the class of non-basic attributes.

2.3. The original approach of Aaberge et al. (2019)

Aaberge et al. (2019) took a dual approach to multidimensional deprivation and poverty measurement and defined deprivation in society via an indicator D where

$$D = P - \sum_{j=0}^{P-1} \Gamma(F_j) \tag{5}$$

In (5), P, as before, refers to the number of possible deprivations suffered by individuals, and F_j is defined as $F_j = \sum_{h=0}^{j} f_h$, with f_h the relative frequency of those who have h deprivations. Finally, Γ is a non-negative and non-decreasing continuous function that represents the preferences of the social planner, with $\Gamma(0) = 0$ and $\Gamma(1) = 1$. Since the mean number of deprivations $\bar{d}$ may be expressed as

$$\bar{d} = P - \sum_{j=0}^{P-1} F_j \tag{6}$$

Combining (5) and (6), we derive that

$$D = \bar{d} + \sum_{j=0}^{P-1} F_j - \sum_{j=0}^{P-1} \Gamma(F_j) \tag{7}$$

However, the mean difference Δ of a distribution $F(t)$ may be expressed as (see, Yitzhaki and Schechtman, 2013, p. 16)

$$\Delta = 2 \int F(t)[1 - F(t)]dt \tag{8}$$

Adapting (8) to the case of discrete data and to the distribution of deprivations, we derive that

$$\Delta_{d_i} = 2 \sum_{j=0}^{P} F_j - 2 \sum_{j=0}^{P} \left(F_j\right)^2 = 2 \left[\sum_{j=0}^{P-1} F_j - \sum_{j=0}^{P-1} \left(F_j\right)^2\right] \tag{9}$$

where Δ_{d_i} refers to the mean difference of the deprivations and we recall that $F_P = (F_P)^2 = 1$.

If we assume in (5) that $\Gamma(F_j) = (F_j)^2$, we conclude, using (6) and (9), that in such a case

$$D = \bar{d} + \left(\frac{1}{2}\right) \Delta_{d_i} \tag{10}$$

The case where $\Gamma(F_j) = (F_j)^2$ was indeed discussed by Aaberge et al. (2019).

3. On the derivation of a rank-dependent multidimensional deprivation index when only binary variables are available

3.1. On the extension of Sen' poverty index and poverty gap profiles

3.1.1. On Shorrocks' (1995) extension of the Sen (1976) index

Let n denote the population size, x_i the income of individual i, z the poverty line, and q the number of people with income $x_i \leq z$. Sen (1976) derived axiomatically a poverty index that is expressed as

$$P_{Sen} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2q - 2i + 1) \left(\frac{z - x_i}{z}\right) \tag{11}$$

Defining x_i^* as $x_i^* = Min\{x_i, z\}$, Shorrocks (1995) extended Sen's index and proposed to define a poverty index $P_{Sen-Shorrocks}$ as

$$P_{Sen-Shorrocks} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left(\frac{z - x_i^*}{z}\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n - 2i + 1) \left(\frac{z - x_i}{z}\right) \tag{12}$$

Shorrocks (1995) stressed that P_{Sen} in (11) is not replication invariant, not a continuous function of individual incomes and fails to satisfy the transfer axiom, whereas the $P_{Sen-Shorrocks}$ index is symmetric, replication invariant, monotonic, homogeneous of degree zero in *z* (poverty line) and *x* (income), normalized, continuous and consistent with the transfer axiom.

3.1.2. On poverty gap profiles or the so-called TIP curves

There has also been a graphical representation of unidimensional poverty: plot on the horizontal axis the cumulative relative population frequencies and on the vertical axis the cumulative values of the expression $\left(\frac{1}{n}\right)Max\left\{\left(\frac{z-x_i}{z}\right), 0\right\}$, ranking the individual by increasing income; you then obtain a "poverty gap profile" (Shorrocks, 1995), which is also called TIP curve (Jenkins and Lambert, 1997; 1998a; 1998b). Shorrocks (1995) then proved that the Sen-Shorrocks index is equal to twice the area below the poverty gap profile.

3.2. Multi-dimensional deprivation in the case of dichotomous variables

3.2.1. Deriving deprivation profiles in the multi-dimensional case

Assume n individuals, P dimensions of well-being, and a dichotomous variable a_{ij} equal to 1 if individual *i* has an achievement in domain *j* (e.g., if *j* refers to "having a good health", $a_{ij} = 1$ if individual *i* is in good health, to 0 otherwise). Let a_i be defined as

$$a_i = \sum_{j=1}^{P} w_j \, a_{ij} \tag{13}$$

where w_j is the weight of dimension *j* and $\sum_{j=1}^{P} w_j = 1$.

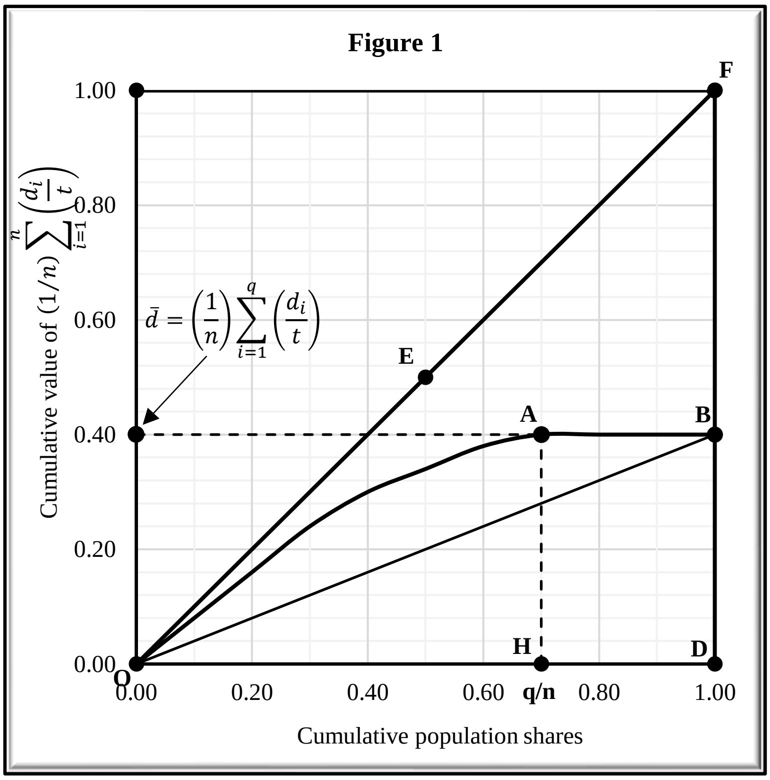
If we define d_{ij} as $d_{ij} = (1 - a_{ij})$, so that $d_{ij} = 1$ if individual *i* is deprived in domain *j*, to 0 otherwise, the weighted deprivation score (d_i) for individual *i* will be expressed as

$$d_i = \sum_{j=1}^{J} w_j d_{ij} \tag{14}$$

The achievement score (a_i) is a "good" so that traditional tools of distributional analysis (e.g., the Lorenz or Generalized Lorenz curves) can be used. But the deprivation score (d_i) is a "bad" (see, Shorrocks, 1998) so that a decrease in an individual's deprivation or in the inequality of the deprivation scores leads to a decrease in "aggregate deprivation".

The concept of poverty gap profile or *TIP* curve previously mentioned may be also applied in the context of multidimensional deprivation. Define an achievement threshold *t*, compute the normalized achievement gaps $\left(\frac{d_i^*}{t}\right) = Max\left\{\left(\frac{t-a_i}{t}\right), 0\right\}$ and then plot on the horizontal axis the cumulative population shares and on the vertical axis the cumulative sum of the expressions $m_i = \left(\frac{1}{n}\right)\sum_{i=1}^{n}\left(\frac{d_i^*}{t}\right) = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{d_i}{t}\right)$, the $d_i^*{}'s$ being ranked by decreasing values. One obtains a rising curve whose slope is non-decreasing and equal to 0 when we reach the (*n* - *q*) individuals with no deprivation (there are *q* individuals with at least one deprivation). The curve is similar to the *TIP* curve previously mentioned.

Note that if $t = 1, \left(\frac{d_i^*}{t}\right) = Max\left\{\left(\frac{1-a_i}{1}\right), 0\right\} \leftrightarrow d_i^* = Max\{(1-a_i), 0\} = Max\{d_i, 0\}$



- In Figure 1 OH refers to the "*prevalence" (P)* or incidence of deprivation [proportion $\left(\frac{q}{n}\right)$ of individuals having some deprivation].

- The slope BOD equals (BD/OD)=$\left[\frac{(1/n)\sum_{i=1}^{n} d_i^*}{1}\right] = \left[\frac{(1/n)\sum_{i=1}^{q} d_i}{1}\right] = \left(\frac{q}{n}\right)\left(\frac{\sum_{i=1}^{q} d_i}{q}\right) = \left(\frac{q}{n}\right)\overline{d_q}$

where $\overline{d_q}$ represents the average percentage of deprivations among those who have at least one deprivation; $\overline{d_q}$ could be labeled the "*breadth" (B)* or intensity of deprivation.

- The curvature of the OA curve indicates the extent of inequality among those deprived in at least one dimension or the *unevenness* (*U*) or inequality of deprivation.

This "deprivation curve" (OAB) is actually an adaptation of the *TIP* curve to multidimensional deprivation with dichotomous variables; given that this curve takes into account the "prevalence", the "unevenness" and the "breadth" of deprivation, we propose to call it the "PUB curve".[1]

3.2.2. Deriving a rank-dependent multidimensional deprivation index (MDI) when only binary variables are available

Like Shorrocks (1995) showed for the uni-dimensional case, it is possible to show that twice the OABDHO area is equal to a Multidimensional Deprivation Index (*MDI*).

More precisely we may write that

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n}(2n-2i+1)\left(\frac{d_i^*}{t}\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q}(2n-2i+1)\left(\frac{d_i}{t}\right) \tag{15}$$

With a union approach (an individual is deprived even if in only one domain), $t = 1$ and then

$$MDI_{union} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n}(2n-2i+1)d_i^* = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q}(2n-2i+1)d_i \tag{16}$$

Using (15), the contribution $(Cont_i)$ of individual *i* to the overall deprivation is expressed as

$$Cont_i = 2\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{d_i}{t}\right)\left[\left(\frac{2n+1}{2}\right) - i\right] \tag{17}$$

Following Shorrocks' (1995), it is easy to show that

$$MDI = \bar{d}\left(1 + G_{d_i}\right) = \bar{d}\left[1 + \left(\frac{d_{EQ}-\bar{d}}{\bar{d}}\right)\right] = d_{EQ} = \bar{d} + \left(\frac{1}{2}\right)\Delta_{d_i} \tag{18}$$

[1] Lasso de la Vega (2010) also introduced deprivation curves derived from deprivation counts and called them the *FD* and the *SD* curves. The focus of the *FD* curve is on the multidimensional headcount ratio, while the *SD* curve shows on the same graph the "headcount ratio, the adjusted headcount ratio, and the average deprivation share according to Alkire and Foster (2007)" (p. 156). The *PUB* curve is very similar to the *SD* curve, but, in addition, it emphasizes the third "I" of multidimensional deprivation: inequality ("unevenness"). Furthermore, it should be observed that Alkire and Foster's methodology (2007, 2011) from which the *FD* and *SD* curves of Lasso de la Vega (2010) are derived, pays no attention to the deprivation distribution and is hence insensitive to the extent of inequality among the multidimensionally poor people.

where $\bar{d}$ and G_{di} are respectively the average level of deprivation and the Gini index of the deprivation scores in the whole population (including those who have no deprivation), $\Delta_{d_i} = 2\bar{d}G_{d_i}$ is the mean difference of the deprivations and d_{EQ} is the "equally distributed equivalent deprivation score". [2]

We may observe that expressions (10) and (18) are identical so that the *MDI* is a specific case of the deprivation measure of Aaberge et al. (2019), that where $\Gamma(F_k) = (F_k)^2$.

Rather than using the traditional Gini index G_{d_i}, one can also use the generalized Gini index that was introduced by Donaldson and Weymark (1980) and apply it to the deprivation scores. The "equally distributed equivalent deprivation score" $d_{EQ,GEN}$ in such a case uses the concept of "ill-fare ranking" (Donaldson and Weymark, 1980) so that

$$d_{EQ,GEN} = \sum_{i=1}^{n} \left(\frac{i^{\beta}-(i-1)^{\beta}}{n^{\beta}}\right) d_i \quad \text{with } 0 \leq \beta \leq 1 \text{ and evidently } d_1 \geq \cdots \geq d_q \geq \cdots \geq 0. \quad (19)$$

In case of tied ranks, we can apply the procedure described in Deutsch and Silber (2005) in the case of occupational segregation.[3]

3.3. Estimating the contribution of different population subgroups to the MDI

Assume K population subgroups, each subgroup k with n_k individuals. Using (15) we write

$$MDI = \left(\frac{1}{n}\right)^2 2\sum_{k=1}^{K}\sum_{i\in k}\left(\frac{d_i}{t}\right)\left[\left(\frac{2n+1}{2}\right) - i\right] \quad (20)$$

i being the ranking of the individual in the whole population and not in his/her subgroup.

[2] It is well known that the Gini index of incomes I_G, like several other income inequality indices that can be related to a welfare function, may be expressed as $I_G = (\bar{y} - y_E)/\bar{y}$, where $\bar{y}$ refers to the average income and y_E to Atkinson's (1970) concept of "equally distributed equivalent level of income" applied to the Gini welfare function. While income is a "good", deprivation is a "bad" so that the Gini index of the deprivation scores is defined as $G_{d_i} = \frac{(d_{EQ}-\bar{d})}{\bar{d}}$.

[3] Let us rank the deprivation scores d_i by decreasing values. Call f_i the population frequency of deprivation score d_i and s_i the share of deprivation score d_i in the total amount of deprivation in the society. Define a variable a_i as $a_i = \left(\sum_{j=1}^{i} f_j\right)^{\beta} - \left(\sum_{j=1}^{i-1} f_j\right)^{\beta}$. Deutsch and Silber (2005) have then shown that the generalized Gini index may I_{GG} be expressed as I_{GG}=1-$\left[\sum_i a_i\left(\frac{s_i}{f_i}\right)\right]$. A similar procedure may be applied to the index *MDI.*

The contribution C_k of population subgroup *k* to multidimensional deprivation is hence

$$C_k = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) 2 \sum_{i \in k} \left(\frac{d_i}{t}\right) \left[\left(\frac{2n+1}{2}\right) - i\right] \quad (21)$$

3.4. Making assumptions concerning the weight of the different deprivation domains

Let *j* refer to a given deprivation domain with $j = 1$ to J. Combining (14) and (15), we derive

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \sum_{j=1}^{J} \frac{w_j d_{ij}}{t} (2n - 2i + 1) \quad (22)$$

so that the contribution $CONTR_j$ of deprivation domain *j* to the overall deprivation becomes

$$CONTR_j = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} w_j \left(\frac{d_{ij}}{t}\right) (2n - 2i + 1) \quad (23)$$

There are quite a few possibilities as far as the choice of the weights w_j of the various dimensions are concerned. In a recent paper Dutta et al. (2021) have however shown that endogenous (data driven) weights violate key properties of poverty indices, namely monotonicity and subgroup consisteny. They hence recommend using exogenous weights, the simplest case being that where all the deprivation domains have the same weight. We will make this assumption so that we rewrite (23) as

$$CONTR_j = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \left(\frac{1}{J}\right) \left(\frac{d_{ij}}{t}\right) (2n - 2i + 1) \quad (24)$$

3.5. Comparing the approach of Chakravarty and D'Ambrosio with that of the *MDI*

There is a clear parallelism between expressions (1) and (16). In (1) deprivation in society is defined as the arithmetic average of the individual deprivations, each individual deprivation d_i being a function of the percentage of possible deprivations individual *i* suffers from. When the parameter α is equal to 2, for example, this individual deprivation is not only higher, the higher the number of domains in which the individual is deprived; this individual deprivation also increases at an increasing rate with the number of deprivations suffered.

In expression (16), deprivation in society is a weighted average of the individual deprivations. Here the individual deprivatiopn d_i is simply a weighted or unweighted average of the number of deprivation domains in which the individual is deprived. But the weight of each individual deprivation d_i is higher, the higher the number of deprivations, hence the term "rank-dependent multidimensional deprivation index" that appears in the title of our paper. In expression (16), these weights increase in a linear way but in expression (19) the parameter β may be chosen in such a way that the weights increase at an increasing rate.

4. Properties of the MDI

As stressed previously, the *MDI* is an extension of the Sen-Shorrocks poverty index applied to the weighted deprivation scores d_i. Therefore, all the properties of the Sen-Shorrocks index stated by Shorrocks (1995) and mentioned previously hold also for the *MDI*.

Alkire and Foster (2016) have stated that the properties of multidimensional poverty methodologies can be classified into three categories: *invariance, subgroup* and *dominance properties*.

Invariance properties include those of *symmetry*, *replication invariance*, *deprivation focus* and *poverty focus*.

4.1. Invariance properties

Symmetry

The reference here is to permutations of achievement vectors across individuals. As stressed by Shorrocks (1995), the Sen-Shorrocks poverty index has this property.

Population replication

Assume a "cloning" of the whole population so that the total population and the number of deprived individuals are now respectively equal to (λn) and (λq) with λ an integer greater than 1. We assume no change in the number of deprivation dimensions. In addition, any deprived individual i with a deprivation score d_i will be replaced by λ individuals with this deprivation score d_i. Here again, Shorrocks (1995) stated that such a property holds for the Sen-Shorrocks poverty index.

Poverty focus

This assumption says that an increment in the achievement of a non-deprived person, that is, of an individual who is not deprived in any dimension, will not affect the value of the multi-dimensional deprivation index (*MDI*). This should be clear from equations (15) and (16) since the *MDI* is only a function of the deprivation of the deprived individuals.

Deprivation focus

This property assumes that the multi-dimensional deprivation index (*MDI*) will be invariant to an increment in a non-deprived achievement. It is easy to check this property too, since if an individual *i* improves his/her achievement in a dimension *j* in which he/she was not deprived, the value of the dichotomous variable d_{ij} will not vary and remain equal to 0.

4.2. Subgroup properties

Alkire and Foster (2016) have mentioned the properties of subgroup consistency and subgroup decomposability.

Subgroup decomposability

The expression for the contribution of subgroup *k* to the overall deprivation (MDI) is given in (21). Combining (20) and (21), we conclude that

$$MDI = \sum_{k=1}^{K} C_k \quad (25)$$

We can therefore compute the contribution of each subgroup to the overall level of deprivation. Note however that C_k in (22) is not identical to what would be the definition of an *MDI* limited to group *k*. This is so because the coefficient $\left[\left(\frac{2n+1}{2}\right) - i\right]$ associated to the deprivation component $\left(\frac{d_i}{t}\right)$ of individual *i* depends on the rank of individual *i* in the whole population, and not in subgroup *k*. A subgroup decomposable deprivation index would be expressed as the sum of a between and a within groups deprivations. But this is not what (21) is expressing. Therefore, we cannot conclude that the multidimensional deprivation index

(*MDI*) is subgroup decomposable in the traditional interpretation of such a breakdown. This is also the case of the Gini index since it is well known that, as soon as there is some overlap between the population subgroups, the decomposition of the Gini index will include three components: a between and a within groups inequality but also a residual which has been shown to be a measure of the overlap between the different distributions (see, for example, Silber, 1989).

It is however possible to take an alternative view of the breakdown of the *MDI* by population subgroups. To derive such an alternative decomposition, we borrow ideas from the literature on alternative decompositions of the Gini index. Deutsch and Silber (1999) have indicated that there is no unique way of decomposing inequality by population subgroups. They mention a decomposition of the Gini index, originally proposed by Lerman and Yizthaki (1991) and Sastry and Kelkar (1994), where the Gini index turns out to be the sum of a between and within groups components, but these two components are not defined in the traditional way. The idea is to keep the original ranking of the individuals, when computing these between and within group components. This idea may be also applied to the breakdown of the *MDI* into a between and a within groups components.

The alternative between groups *MDI* is then defined as

$$MDI_{BETWEEN}^{Alternative} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left(\frac{\bar{d}_i}{t}\right) \tag{26}$$

where *i* refers to the original rank of an individual and $\bar{d}_i$ refers to the average deprivation level in the population subgroup to which individual *i* belongs.

The alternative within groups component is then expressed as

$$MDI_{WITHIN}^{Alternative} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left[\frac{(d_i - \bar{d}_i)}{t}\right] \tag{27}$$

In Appendix A, we give a simple empirical illustration of what we called the traditional and the alternative decompositions of the *MDI*. Figure A.1 gives a graphical representation of the alternative decomposition.

In short, when using the alternative approach, it is possible to affirm that the *MDI* is decomposable by population subgroups.[4]

Subgroup consistency

Shorrocks (1995, p. 1226) stressed that, like the Sen poverty index (P_{SEN}), the Sen-Shorrocks poverty index ($P_{Sen-Shorrocks}$) is not subgroup consistent, but "it is an ideal measure of poverty in all other respects". Since the *MDI* is equivalent to the $P_{Sen-Shorrocks}$ index, but applied to multidimensional deprivation, we conclude that the *MDI* is not subgroup consistent.

Dimensional breakdown

The dimensional breakdown or factor decomposability property technically requires that "after identification has taken place" and the poverty status of each person has been fixed, multidimensional poverty can be expressed as a "weighted sum of dimensional components" (Alkire and Foster, 2019, p. 13). This implies, following Chakravarty et al. (1998), that the overall poverty index is a weighted sum of the poverty measures of the various dimensions, these measures being only function of the distribution of the individual achievements in the corresponding dimension and of the threshold selected for this dimension.

This is however not the case for *MDI*, since the individual level weight ($2n - 2i + 1$) we use to compute each dimensional component depends on the overall ranks of the poor, so any change in the joint distribution is likely to change the rank of the individuals and thus the value of each component. However, in the traditional decomposition of the Gini index by income sources, it is generally stated that each source's contribution is "the product of its own inequality, its share of total income, and its correlation with the rank of total income" (Lerman and Yitzhaki, 1985, p. 153).[5] Now clearly the poverty dimensions in a multidimensional framework play the role of the income sources in a unidimensional analysis of inequality; therefore, while it cannot be said that our *MDI* has the property of dimensional breakdown in the way Chakravarty et al. (1998) and Alkire and Foster (2019) define this feature, we can at

[4] One may however wonder how convenient this alternative approach is for policy purposes. It has been pointed to us that India recently released its computations of multidimensional poverty for more than 600 districts. Our approach suggests that the population should be ranked at the country level and then those ranks should be used at all subgroup level. For a big country like India, this may indeed not very easy to implement, if census data rather than surveys are used.

[5] See also, Fei, Ranis and Kuo (1978) for a previous presentation of the decomposition of the Gini index by factor components, that is, by income sources.

least state that when using the *MDI* to measure (multidimensional) poverty, we can compute the contribution of each dimension to the overall value of the *MDI*.

4.3. Dominance

Alkire and Foster (2016) included here two properties. There is first the concept of Weak Monotonicity according to which an increase in the achievement of an individual cannot increase deprivation. Then, there is the notion of Weak Rearrangement that requires that a progressive transfer among the deprived individuals, which is the consequence of an "association-decreasing rearrangement", cannot increase deprivation.

Monotonicity

Shorrocks (1995) stated that the index $P_{Sen-Shorrocks}$ is monotonic. We can therefore conclude that the MDI has the property of monotonicity.

Transfers

Let us first state that in the context of uni-dimensional poverty measurement Shorrocks (1995) stressed that the $P_{Sen-Shorrocks}$ index is consistent with the transfer axiom. When applying this property to multidimensional deprivation analysis, we can therefore conclude that if, within a given deprivation domain *j*, a transfer takes place from a more to a less deprived individual, assuming no change in the ranking of the individuals, the *MDI* will decrease. More precisely, assume that originally individual *i,* as a whole, was more deprived than individual *m* and was deprived in domain *j* while individual *m* was not. After the "transfer" individual *i* remains more deprived than individual *m,* but he/she has one deprivation less, while individual *m* has one more deprivation than originally. In such a case the *MDI* will decrease.

The same kind of reasoning applies when a transfer takes place between individuals and across domains. Assume, for example, that individual *h* has n_h deprivations and that individual *i* has n_i deprivations with $n_h > n_i$, that individual *h* is deprived in domain *j* but not in domain *k* and individual *i* in domain *k* but not in domain *j*. If, for some reason, a change occurs such that individual *h* is not deprived any more in domain *j* while individual *i* who was deprived in domain *k* becomes also deprived in domain *j*. Assume, however, that, after such a "transfer" of deprivations, individual *h* has still more deprivations than individual *i*. If we assume that all the domains have the same weight, it is easy to observe, using (23) that the *MDI* will decrease.

Given that in the formulation of the *MDI* in (23), which refers to the case of equal weights, only the number of deprivations of each individual is taken into account, no matter in which domains these deprivations take place, the notion of "*Weak Dimensional Rearrangement among the deprived individuals"*, which was discussed by Alkire and Foster (2016), is not relevant.

Rather than analyzing the impact of a transfer of deprivations between two individuals *h* and *i*, let us assume that these two individuals switch their deprivations. In other words, using the example given previously, we would observe that in the new situation individual *h* is deprived in domain *k* but not in domain *j* and individual *i* in domain *j* but not in domain *k*. Clearly such a switch will not affect the number of deprivations of each individual and hence there will be no change in the value of the *MDI*.

In defining the *MDI* in (23), which refers to the case of equal weights for the different deprivation domains, we made the assumption that the various deprivation domains are perfect substitutes.

The case is different when examining the case of unequal weights. It should be clear that even in the case where the various dimensions have different weights, a transfer of deprivations between two individuals of the kind described above, whether it takes place within a given domain or across domains, will lead to a decrease in the *MDI*, as long as the ranking of the individuals by the number of deprivations they suffer from, is not affected. However when the deprivation domains have not the same weight, the switch of deprivations between two individuals and two domains with unequal weights, will lead either to an increase or a decrease in the value of the *MDI*, depending on the assumption made concerning the weights of domains *j* and *k*.

4.4. Comparing deprivation profiles and comparing *MDI* indices

Lasso de la Vega (2010) has proven the equivalence between dominance of one *SD* curve over another and the values of the corresponding multidimensional poverty measures *MP* that she defined and that were assumed to obey the following five axioms:

- Poverty focus: the multidimensional poverty measure *MP* remains unchanged if the poverty score of an individual defined as "overall non-poor" decreases.
- Dimensional monotonicity: the multidimensional poverty measure *MP* will decrease if the poverty score of any individual defined as "overall poor" decreases.
- Symmetry: No other characteristic, except the number of weighted dimensions in which an individual is deprived, will affect the multidimensional poverty measure *MP*.
- Replication Invariance: A "cloning" of the deprivation vector of all the individuals will not affect the multidimensional poverty measure *MP*.
- Distribution sensitivity: A decrease in poverty, due to a decrease in the poverty score of a poor individual, should be greater, the higher the poverty score of this individual.

In other words, when the *SD* curve of a deprivation vector d' lies above the *SD* curve of a deprivation vector *d* with the same or different population sizes, any poverty measure having the five properties listed above will rank in the same way these two deprivation vectors.

Note that the dimension adjusted headcount ratio (the ratio of the number of weighted deprivations suffered by those defined as "overall deprived" and the total (maximum) number of weighted deprivations) introduced by Alkire and Foster (2011) violates the distribution sensitivity axiom. Lasso de la Vega proved however that if two deprivation vectors (corresponding to two different societies) can be unanimously ranked by the dimension adjusted headcount ratio, whatever the value of the dimension cutoff, then all poverty counting measures satisfying the property of distribution sensitivity will rank societies in the same way. Lasso de la Vega (2010) also examined the case of intersecting *SD* curves and showed that it is possible to obtain robust conclusions provided one restricts the set of identification cutoffs.

The question is whether we can find a similar correspondence between the ranking of *SD* curves and the *MDI*. In fact the ordinal approach to uni-dimensional poverty analysis seems to have been originally introduced by Spencer and Fisher (1992). Jenkins and Lambert (1997, p. 317) then introduced the concept of *TIP* ("Three *I*'s of Poverty") curve. Subsequently, Jenkins and Lambert (1998b, p. 47) stated in their Theorem 3 that "given any two income distributions *x* and *y* and poverty lines z_x and z_y, *TIP* dominance of the normalized poverty gap distribution Γ_y over the normalized poverty gap distribution Γ_x is necessary and sufficient to ensure $Q(x \mid k.z_x) \leq Q(y \mid k.z_y)$ for all $k \in (0,1]$ and for all poverty measures $Q \in \mathbf{Q}$", the latter being replication invariant and increasing Schur-convex functions of the normalized gaps.

These deprivation profiles or *TIP* curves may naturally be used when adopting the $P_{Shorrocks-Sen}$ rather than the P_{Sen} index, as shown in Shorrocks (1995).

The *MDI* introduced in the present paper is an extension of the $P_{Sen-Shorrocks}$ index to the case of multidimensional deprivation. Moreover, we have mentioned previously that Lasso de la Vega's *SD* curve is a simple adaptation of the notion of *TIP* curve to the multidimensional case, when one assumes that deprivation in a given domain is only measured via dichotomous variables. It seems therefore that one could apply the theorem of Jenkins and Lambert (1998b) stated previously, provided the deprivation profiles of the distributions we compare, do not intersect.[6]

5. A simple empirical illustration

In this section, we present an empirical illustration of the *PUB* curve, the *MDI* and its decomposition by deprivation indicator, using data from four Central American countries, namely, Guatemala, El Salvador, Honduras, and Nicaragua (for previous work on multidimensional poverty in these countries, using other approaches, see Espinoza-Delgado and Silber, 2018, 2021).

To estimate multidimensional poverty in these Central American countries, we used data from the Guatemala National Survey of Living Conditions (2014) (GUA-ENCOVI2014), the El Salvador Multipurpose Household Survey (2016) (ELS-EHPM2016), the Honduras Multipurpose Household Survey (2013) (HON-EPHPM2013), and the Nicaragua National Household Survey on Living Standards Measurement (2014) (NIC-EMNV2014), which are nationally representative. In our exercise, we focus on individuals who are between 18 and 59 years old, are identified as household members and completed a full interview; in other words, we use the individual, rather than the household, as the unit of analysis and focus on the adult

[6] In a recent paper, Azpitarte et al. (2020) introduced fundamental conditions whose fulfilment is both necessary and sufficient to ensure that poverty comparisons are robust to changes in individual poverty functions, dimensional weights and poverty cut-off. As stated by the authors, these conditions may be cumbersome when the number of variables is large. This is the reason why they also derive conditions whose fulfilment is necessary, but insufficient for robust first- and second-order poverty comparisons. The extension of the Sen-Shorrocks index to multidimensional poverty proposed in the present paper might be a simpler way of analyzing dominance.

members of the households, approximately 50% of the population in the countries studied (from a low of 47.7% in Honduras up to a maximum of 59.3% in El Salvador).

Regarding the empirical design of the *MDI*, we considered five deprivation dimensions (education, employment, water and sanitation, energy and electricity, and the quality of the dwelling) with ten indicators, which are certainly among the most significant aspects of individual well-being (Stiglitz et al., 2009a, 2009b). The specific indicators chosen for each of the five dimensions and the corresponding deprivation definitions are presented in Table 1; this table also shows the weighting structure that we used: equal-nested weights.

Table 1: Dimensions [in parenthesis the related Sustainable Development Goal (SDG)], indicators, weights, and deprivation cut-offs

Dimensions	Indicators	Weights (%)	Deprivation indicators: He / She is deprived if he / she…
1. Education (Goal 4 of the SDGs)	1.1. Schooling achievement	20	has not completed lower secondary school (nine years of schooling approximately).
2. Employment (Goal 8 of the SDGs)	2.1. Employment status	20	is unemployed, employed without pay, or a discouraged worker or a domestic worker or an unpaid care worker who reported that he/she "did not have a job" but was available to work.
3. Water and sanitation (Goal 6 of the SDGs)	3.1. Improved water source	10	does not have access to an improved water source or has access to it, but out of the house and yard/plot.
	3.2. Improved sanitation	10	only has access to an unimproved sanitation facility (a toilet or latrine without treatment or a toilet flushed without treatment to a river or a ravine) or to a shared toilet facility.
4. Energy and electricity (Goal 7 of the SDGs)	4.1. Type of cooking fuel	10	is living in a household which uses wood and/or coal and/or dung as main cooking fuel.
	4.2. Access to electricity	10	does not have access to electricity.
5. Quality of dwelling (Goal 11 of the SDGs)	5.1. Housing materials	5	is living in a house with dirt floor and/or precarious roof (waste, straw, palm and similar, other precarious material) and/or precarious wall materials (waste, cardboard, tin, cane, palm, straw, other precarious material).
	5.2. People-per-bedroom	5	has to share a bedroom with two or more people.
	5.3. Housing tenure	5	is living in an illegally occupied house or in a borrowed house.
	5.4. Assets	5	does not have access to more than one durable good of a list that includes: Radio, TV, Refrigerator, Motorbike, Car.

The PUB curve: prevalence (P), unevenness (U) and deprivation breadth (B) curve

We assumed that the threshold t was equal to 1, that is, we take a union approach. Figure 2 displays the *PUB* curve for Guatemala, El Salvador, Honduras, Nicaragua, and Central America as a whole; in this figure, the cumulative population frequencies are plotted on the X-axis, while the cumulative values of $(1/n)\sum_{i=1}^{n}(d_i/1)$ are plotted on the Y-axis. Overall, the left side of Figure 2 suggests that in the Central American region, the highest and lowest levels of multidimensional poverty are found in Guatemala and El Salvador, respectively. The *PUB* curve of Honduras dominates that of El Salvador, so that multidimensional poverty in the former country is always higher than in the latter, regardless of the population decile we choose. The cases of the Guatemalan and Nicaraguan curves are interesting. Figure 2 shows that the Nicaraguan curve crosses the Guatemalan curve once from above around the 25% point on the horizontal axis (see the right side of the figure), suggesting that overall multidimensional poverty is higher in Guatemala than in Nicaragua only from this point on, i.e., the poorest of the poor are in Nicaragua.

Figure 2: "PUB Curves" for Central American as a whole (CA), Guatemala (GUA), El Salvador (SAL), Honduras (HON), and Nicaragua (NIC). *Sources*: Authors' estimates based on GUA-ENCOVI2014, ELS-EHPM2016, HON-EPHPM2013, and NICEMNV2014.

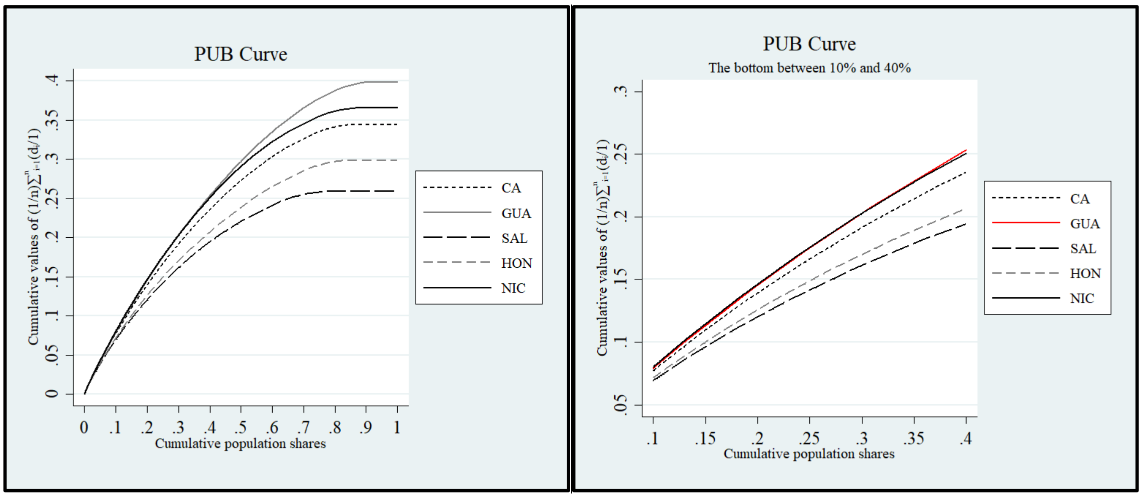


Table 2: Absolute and relative contributions of each indicator to the overall MDI. *Sources*: Authors' estimates based on GUA-ENCOVI2014, ELS-EHPM2016, HON-EPHPM2013, and NIC-EMNV2014.

Guatemala											
Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Housing tenure**	**Assets**	MDI
Absolute	0.2645	0.0664	0.0471	0.1107	0.1444	0.0360	0.0309	0.0495	0.0103	0.0357	0.7956
Relative	33.2%	8.3%	5.9%	13.9%	18.2%	4.5%	3.9%	6.2%	1.3%	4.5%	100.0%
El Salvador											
Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Housing tenure**	**Assets**	MDI
Absolute	0.1736	0.0726	0.0417	0.0849	0.0204	0.0256	0.0184	0.0450	0.0180	0.0167	0.5168
Relative	33.6%	14.0%	8.1%	16.4%	3.9%	4.9%	3.6%	8.7%	3.5%	3.2%	100.0%
Honduras											
Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Housing tenure**	**Assets**	MDI
Absolute	0.2365	0.0644	0.0247	0.0466	0.1100	0.0242	0.0177	0.0439	0.0056	0.0233	0.5969
Relative	39.6%	10.8%	4.1%	7.8%	18.4%	4.1%	3.0%	7.3%	0.9%	3.9%	100.0%
Nicaragua											
Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Housing tenure**	**Assets**	MDI
Absolute	0.2247	0.0786	0.0676	0.0860	0.1051	0.0265	0.0394	0.0535	0.0157	0.0332	0.7303
Relative	30.8%	10.8%	9.3%	11.8%	14.4%	3.6%	5.4%	7.3%	2.2%	4.5%	100.0%
Central America as a whole											
Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Housing tenure**	**Assets**	MDI
Absolute	0.2342	0.0693	0.0448	0.0876	0.1066	0.0298	0.0272	0.0481	0.0117	0.0290	0.6883
Relative	34.0%	10.1%	6.5%	12.7%	15.5%	4.3%	3.9%	7.0%	1.7%	4.2%	100.0%

Table 2 illustrates the contribution of the different domains to the overall deprivation for the case of Guatemala, El Salvador, Honduras, and Nicaragua, as well as for Central American as a whole. Table 2 presents the absolute and relative contributions to the overall estimate of multidimensional poverty of each of the ten indicators used to measure multidimensional poverty in Central America; the overall estimates are shown in the last column of the table. The table indicates that in Central America, education is the largest contributor to multidimensional poverty; deprivations in this dimension accounts for one-third of the estimated MDI in each of the countries.

6. Concluding remarks

In this paper, we have introduced a new multidimensional deprivation index (*MDI*) that is an extension of the Sen-Shorrocks index of unidimensional poverty to the multidimensional case. Interestingly, it turns out that the *MDI* is a particular case of a measure of multidimensional deprivation recently introduced by Aaberge et al. (2019). In addition, by linking the *MDI* to the Sen-Shorrocks index, we have been able to derive a simple graphical representation that takes into account the prevalence (incidence), unevenness (inequality) and breadth (intensity) of deprivation. This curve is an extension of the *TIP* curve of Jenkins and Lambert (1997) to the multidimensional case and turns out to be very similar to the *SD* curve introduced by Lasso de la Vega (2010), although based on a different approach. It is therefore possible to compare the deprivation profiles of two or more countries, or of a country during various periods, and to derive dominance relationships. The *MDI* can be broken down by population subgroup, although it is not a subgroup consistent index, but "it is an ideal measure of poverty in all other respects" (Shorrocks, 1995, p. 1226). We also showed that while the *MDI* does not have the property of dimensional breakdown, we can compute the contribution of each domain to the overall deprivation, in the same way as in the literature on the Gini index, one can compute the contribution of each income source to the Gini index or income inequality. These two decompositions, which may be considered as not standard, should allow policy makers to detect the population subgroups and the deprivation domains that require special attention. The empirical illustration of the paper, which looked at four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua), allowed us to conclude that education is the largest contributor to multidimensional deprivation, since it accounts for one-third of the *MDI* in each of the countries.

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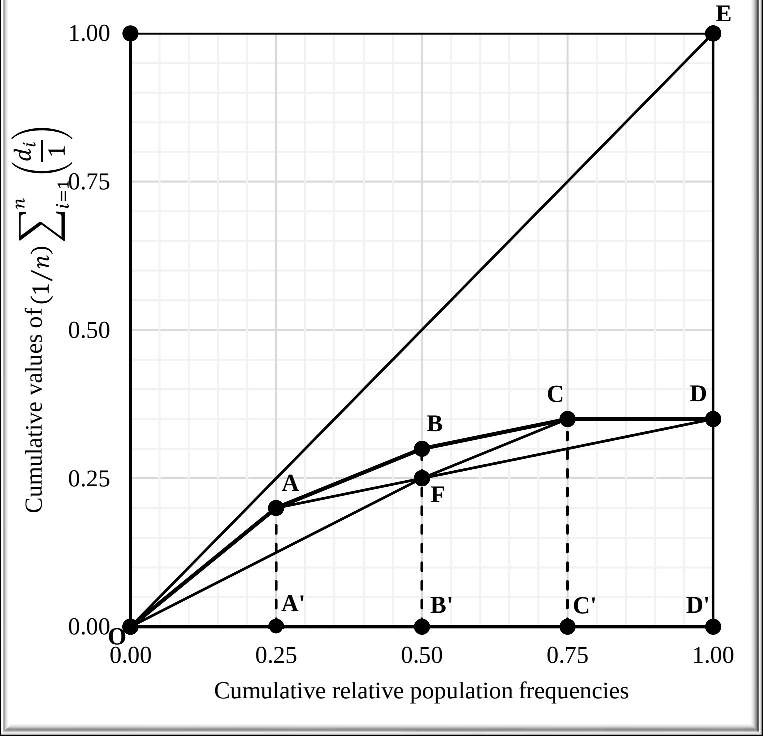
Appendix

Appendix A: The decomposition of the MDI by population subgroups

Assume a population of four individuals. Three of them have a certain number of deprivations and one is without any deprivation so that $n = 4$ and $q = 3$. Suppose that there are 5 domains of deprivation ($j = 1$ to 5). Individual 1 is deprived in domains 1, 2, 4, 5 so that $\left(d_1 = \left(\frac{4}{5}\right)\right)$, individual 2 in domains 3 and 4 $\left(d_2 = \left(\frac{2}{5}\right)\right)$ and individual 3 in domain 5 $\left(d_3 = \left(\frac{1}{5}\right)\right)$. Individual 4 has no deprivation. Suppose that individuals 1 and 3 belong to group A and individuals 2 and 4 to group B. Let us also assume that the threshold t is equal to 1. Finally define p_i as that $p_i = \frac{1}{n} d_i = \frac{1}{4} d_i$. Figure A-1 illustrates this case.

Figure A-1



Using (20) the *MDI* is expressed as

$$\text{MDI} = \left(\frac{1}{16}\right)\{[(7)(0.8)] + [(5)(0.4)] + [(3)(0.2)]\} = \frac{(5.6+2+0.6)}{16} = \frac{8.2}{16}$$

Using (27) we then derive that the contributions C_A and C_B of groups A and B are expressed as

$C_A = \left(\frac{1}{16}\right)\{[(7)(0.8)] + [(3)(0.2)]\} = \frac{6.2}{16}$

$C_B = \left(\frac{1}{16}\right)\{[(5)(0.4)]\} = \frac{2}{16}$

It is easy to observe that, as expected, the sum of these two contributions is equal to $\frac{6.2+2}{16} = \frac{8.2}{16}$, which is the value of the *MDI* for the whole population.

The graphical representation of a traditional decomposition

In Figure A-1 the curve *OABCD* represents what we previously called the *PUB* curve. The line OE is the deprivation curve that would be obtained if everyone had the same and maximal level of deprivation, namely (5/5) so that the height ED' is, as expected, equal to $4\left(\frac{1}{4}\right)\left(\frac{5}{5}\right) = 1.$ It is easy to check that the heights *AA', BB', CC'* and *DD* are respectively equal to 0.2, 0.3, 0.35 and 0.35 and that the areas OAA', AA'B'B, BB'C'C and CC'D'D are respectively equal to 0.025, 0.0625, 0.08125 and 0.0875. The sum of these 4 areas which corresponds to the area OABCDD'C'B'A'O is then equal to 0.25625. Twice this sum gives us 0.5125=$\left(\frac{8.2}{16}\right)$, which is, as expected and shown previously, the value of the MDI when all the domains have the same weight.

Given that individuals 1 and 3 belong to group A and individual 2 and 4 to group B, it is easy to check that the average number of deprivations in group A is $\frac{4+2}{2} = 3$ and in group B it is $\left(\frac{2+0}{2}\right) = 1.$ We can therefore draw in Figure B.1 a broken curve OFD. On the section OF the height of point F corresponds to the total deprivation in group *A* which includes individuals 1 and 3 and hence it is equal to $\left[\left(\frac{1}{4}\right)\left(\frac{4}{5}\right)\right] + \left[\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\right] = \left(\frac{5}{20}\right) = 0.25.$ Similarly, the difference between the height of point D and that of point F corresponds to the deprivation in group B and is hence expressed as $\left[\left(\frac{1}{4}\right)\left(\frac{2}{5}\right)\right] + \left[\left(\frac{1}{4}\right)\left(\frac{0}{5}\right)\right] = \left(\frac{2}{20}\right)$ = 0.1. The height of point D is therefore 0.25+0.1=0.35. The area below the curve OFDD'O is therefore, computed as $\left[\left(\frac{1}{2}\right)(0.5)(0.25)\right] + \left\{\left(\frac{1}{2}\right)(0.5)[0.25 + 0.35]\right\} = 0.0625 + 0.150 = 0.2125.$ Twice this area, that is, 0.425, is hence the between groups *A* and *B* components of multidimensional deprivation.

We can also compute the within groups A and B components of multidimensional deprivation. The within group *A* deprivation is evidently the area OAF while that within group *B* is the area FCD. Now OAF=[(OAA')+(AA'B'F)]-(OFB') with OAA'=$\left[\left(\frac{1}{2}\right)(0.25)(0.2)\right] = 0.025;$ AA'B'F=$\left[\left(\frac{1}{2}\right)(0.25)(0.2 + 0.25)\right] = 0.05625; OFB' = [(0.5)(0.5)(0.25)] = 0.0625.$ We therefore derive that the area OAF is equal to (0.025+0.05625)-0.0625= 0.01875. Twice this number gives us the within group A multidimensional deprivation and it is equal to 0.0375.

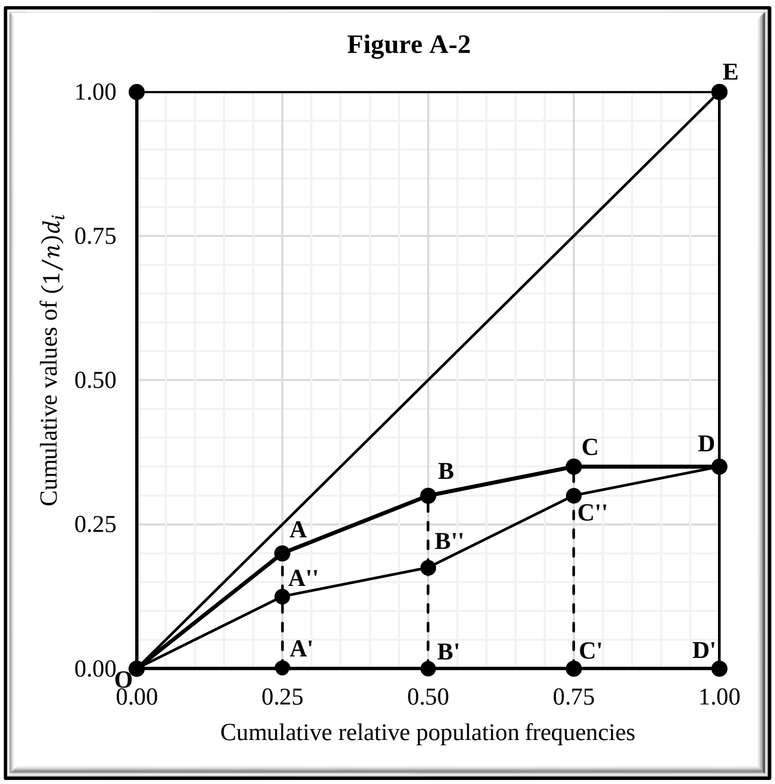
The within group B deprivation is given by the triangle FCD whose area is equal to $[(FB'C'C + CC'D'D) - FB'D'D]$. But FB'C'C=$\left(\frac{1}{2}\right)(0.25)(0.25+035)$=0.075; CC'D'D = $(0.25 \times 0.35) = 0.0875$; and $\mathrm{FB'D'D} = \left(\frac{1}{2}\right)(0.5)(0.25+0.35) = 0.15$. The area FCD is hence equal to (0.075+0.0875)-0.15=0.0125. Twice this area is therefore equal to the within group B multidimensional deprivation, that is, to 0.025.

Let us now compute the area ABCF that corresponds to the overlap between group *A* and group *B.* We may write that ABCF = (AA'B'B+BB'C'C)-(AA'B'F+FB'C'C). AA'B'B=$(0.5)(0.25)(0.2+0.3) = 0.0625$; $\mathrm{BB'C'C} = (0.5)(0.25)(0.3+0.35) = 0.08125$; $\mathrm{AA'B'F} = (0.5)(0.25)(0.2+0.25) = 0.05625$; FB'C'C=$(0.5)(0.25)(0.25+0.35) = 0.075$. Therefore, ABCF = (0.0625+0.08125) - (0.05625+0.075) = 0.0125. Twice this area will be the overlap component of the MDI and it is equal to 0.025.

The sum of the three components (between groups, within groups and overlap deprivation) is then equal to (0.425+0.0375+0.025+0.025) = 0.5125 = $\left(\frac{8.2}{16}\right)$ = *MDI.*

The graphical representation of an alternative decomposition of the MDI

Figure A-2 gives a graphical representation of this alternative decomposition.



As in Figure A-1 the curve ABCD represents the actual *PUB* curve, and it is drawn by ranking the individuals by decreasing level of deprivation This ranking will be kept when drawing the deprivation curve that would be observed if each individual's deprivation was the average deprivation of the group to which he/she belongs. We saw previously that the average deprivation in group *A*, which includes individuals 1 and 3, is (4+1)/2=2.5 while the average deprivation in group B is (2+0)/2=1. Keeping the original ranking of the individual we conclude that the height of point A'' which corresponds to this deprivation of individual 1 will be $\left(\frac{1}{4}\right)\left(\frac{2.5}{5}\right)=\left(\frac{2.5}{20}\right)=0.125$. To reach the second point (B'') on this "alternative average deprivation curve" we add to the height of point A' the average deprivation in group *B* (equal to 1) since individual 2 belongs to group *B* so that the height of point B'' is 0.125 +$\left[\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\right]=0.125+0.050=0.175$. The same idea is applied to compute the height of point C". Starting from B" we have to add a height which corresponds to the average deprivation in group A since individual 3 belongs to group A and so the height of point C'' is 0.175+$\left[\left(\frac{1}{4}\right)\left(\frac{2.5}{5}\right)\right]=0.175+0.125=0.3$. Finally, by adding to the height of point C'' a height corresponding to the average deprivation in group *B* (individual 4 belongs to group *B*) we end up with 0.3+$\left[\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\right]=0.3+0.05=0.35$, which is indeed the height of point D. Clearly, the area OA''B''C''DD'O corresponds to half the value of the alternative between groups deprivations while the area OABCDC"B"A"O represents half the value of the within groups deprivation.

It is easy to find out that the area OA''B''C''DD'O is equal to $\{0.5\times0.25\times0.125\}+\{0.5\times0.25\times(0.125+0.175)\}+\{0.5\times0.25\times(0.175+0.3)\}+[0.5\times0.25\times(0.3+0.35)]$ = 0.015625 + 0.0375 + 0.059375 + 0.08125 = 0.19375. Twice this value (0.3875) is hence the value of the alternative between groups deprivation.

This result can also be obtained by applying (22) to the average incomes of the group to which each individual belongs, giving each individual his/her original rank. We then obtain

$$\left(\frac{1}{16}\right)\left\{\left[(7)\left(\frac{2.5}{5}\right)\right]+\left[(5)\left(\frac{1}{5}\right)\right]+\left[(3)\left(\frac{2.5}{5}\right)\right]+\left[(1)\left(\frac{1}{5}\right)\right]\right\}=\left(\frac{1}{80}\right)(17.5+5+7.5+1)=\left(\frac{31}{80}\right)=0.3875$$

The within groups deprivation (the area OABCDC"B"A"O) is computed as

$\{0.5\times0.25\times(0.2-0.125)\}$+$\{0.5\times0.25\times[(0.2-0.125)+(0.3-0.175)]\}+\{0.5\times0.25\times[(0.3-0.175)+(0.35-0.3)]\}$+$\{0.5\times0.25\times[(0.35-0.3)]\}$ = 0.009375 + 0.025 + 0.021875 + 0.00625 = 0.0625. Twice this area is hence equal to 0.125.

This result may be obtained by applying (20) to the difference for each individual between his/her actual deprivation and the average deprivation of the group to which /she belongs, each individual being assigned again his/her original rank. We then get

$$\left(\frac{1}{16}\right)\left\{\left[(7)\left(\frac{4-2.5}{5}\right)\right]+\left[(5)\left(\frac{2-1}{5}\right)\right]+\left[(3)\left(\frac{1-2.5}{5}\right)\right]+\left[(1)\left(\frac{0-1}{5}\right)\right]\right\}=\left(\frac{1}{80}\right)((10.5+5)-(4.5+1))$$

$$=\frac{10}{80}=0.125$$

The sum of these alternative between and within group's deprivation is hence equal to 0.3875+0.125=0.5125=$\frac{8.2}{16}=MDI$.

Appendix B: The *MDI* as a specific case of the deprivation index of Aaberge et al. (2019): a simple illustration

Let us assume that there are 5 individuals and 10 deprivation domains. Each deprivation has the same weight. Table B-1 below indicates how many deprivations each individual has.

Table B-1: A simple numerical illustration

Number k of deprivations	**Number of individuals**	**Relative frequency f_k of deprivations**	**Cumulative relative frequency F_k of deprivations**	$(F_k)^2$	$(1-F_k)$	$(1-F_k)^2$	$\int(1-F_k)$	$\int(1-F_k)^2$
0	1	0.2	0.2	0.04	0.8	0.64	0.8	0.64
1	0	0	0.2	0.04	0.8	0.64	1.6	1.28
2	1	0.2	0.4	0.16	0.6	0.36	2.2	1.64
3	0	0	0.4	0.16	0.6	0.36	2.8	2
4	0	0	0.4	0.16	0.6	0.36	3.4	2.36
5	0	0	0.4	0.16	0.6	0.36	4	2.72
6	1	0.2	0.6	0.36	0.4	0.16	4.4	2.88
7	1	0.2	0.8	0.64	0.2	0.04	4.6	2.92
8	0	0	0.8	0.64	0.2	0.04	4.8	2.96
9	0	0	0.8	0.64	0.2	0.04	5	3
10	1	0.2	1	1	0	0		

The Gini index G_{c_i} of the distribution of the deprivations may then be computed [see, expression (4) in Berrebi and Silber, 1983] as

$$G_{c_i} = \left[\left(\frac{4}{5}\right)\left(\frac{10-0}{25}\right) + \left(\frac{2}{5}\right)\left(\frac{7-2}{25}\right)\right] = \left(\frac{40+10}{125}\right) = 0.4$$

where 25 in the denominator refers to the total number of deprivations in the population and 5 is the number of individuals. Using (15), we conclude that

$$MDI = \bar{c}\left(1 + G_{c_i}\right) = 5(1 + 0.4) = 7$$

Note that it is also possible to compute the G_{c_i} index using the following formulation of the Gini index (see, Yitzhaki and Schechtman, 2013, p. 15):

$$G_{c_i} = 2\{\int[1 - F(k)]dk\} - 2\{\int[1 - F(k)]^2 dk\} \quad \text{(B-1)}$$

Using the data of Table B-1, we conclude that $\left\{\int_0^9 [1 - F(k)]dk\right\} = 5$ and that

$\left\{\int_0^9 [1 - F(k)]^2 dk\right\} = 3$

Using the data of Table B-1, we conclude that $G_{c_i} = 2(5 - 3)\left(\frac{1}{10}\right) = 0.4$

Since the mean difference Δ_{c_i} of the deprivations is expressed (see, Kendall and Stuart, 1969) as

$$\Delta_{c_i} = 2\,\bar{c}G_{c_i} \quad \text{(B-2)}$$

where $\bar{c}$ is the mean number of deprivations, which is here equal to (2+6+7+10)/5=5, we conclude that $\Delta_{c_i} = 2 \times 5 \times 0.4 = 4$.

Aaberge et al. (2019) have suggested using as measure of deprivation in a society an index $D_\Gamma(F)$ defined (see, their expression (2.4)) as

$$D_\Gamma(F) = r - \sum_{k=0}^{r-1} \Gamma(F_k) \quad \text{(B-3)}$$

where *r* refers to the maximum number of deprivation (in our simple illustration *r* = 10) If we take a "union approach" the function Γ has to be convex. A simple convex function would be

$\Gamma(F_k) = (F_k)^2$ so that we end up with

$$D_I(F) = r - \sum_{k=0}^{r-1}(F_k)^2 \quad \text{(B-4)}$$

Using the data of Table C-1, we easily find that $\sum_{k=0}^{r-1}(F_k)^2 = \sum_{k=0}^{9}(F_k)^2 = 3$. Since $r = 10$, we conclude that $D_I(F) = 10 - 3 = 7$.

Appendix C: Decomposition of *MDI* by domains.

Recall that in (19) *MDI* is expressed as

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \sum_{j=1}^{J} \frac{w_j d_{ij}}{t} (2n - 2i + 1) \qquad \text{(C-1)}$$

Assume to simplify equal weights so that $w_j = \left(\frac{1}{J}\right) \forall j$.

In addition take a "union approach" so that $t = 1$.

We can then express (C-1) as

$$MDI = \left(\frac{\sum_{j=1}^{J}\sum_{i=1}^{n} d_{ij}}{nJ}\right) \sum_{j=1}^{J} \left(\frac{\sum_{i=1}^{n} d_{ij}}{\sum_{j=1}^{J}\sum_{i=1}^{n} d_{ij}}\right) \left[\sum_{i=1}^{n} \frac{d_{ij}}{\sum_{i=1}^{n} d_{ij}} \frac{(2n-2i+1)}{n}\right] \qquad \text{(C-2)}$$

Define now b_{ij} as $b_{ij} = \frac{d_{ij}}{\sum_{i=}^{n} d_{ij}}$ so that b_{ij} refers to the share of individual *i* in the total amount of deprivation in the population in domain *j*.

Define also s_j as $s_j = \left(\frac{\sum_{i=1}^{n} d_{ij}}{\sum_{j=1}^{J}\sum_{i=1}^{n} d_{ij}}\right)$. In other words s_j represents the share of domain *j* in the total amount of deprivation in the population (all domains included).

Finally call $\bar{d}$ the ratio $\left(\frac{\sum_{j=1}^{J}\sum_{i=1}^{n} d_{ij}}{nJ}\right)$ so that refers to the average level of deprivation per individual and per domain in the population.

We can now rewrite (C-2) as

$$MDI = \bar{d} \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij} \left(1 + \left(\frac{n-2i+1}{n}\right)\right)\right] \qquad \text{(C3)}$$

$$\leftrightarrow MDI = \left\{(\bar{d}) \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij}\right]\right\} + \bar{d} \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij} \left(\frac{n-2i+1}{n}\right)\right] \qquad \text{(C-4)}$$

It is then easy to check that

$$\left\{(\bar{d}) \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij}\right]\right\} = \bar{d} \qquad \text{(C-5)}$$

so that

$$MDI = \bar{d}\left\{1 + \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij}\left(\frac{n-2i+1}{n}\right)\right]\right\} \qquad \text{(C-6)}$$

While *i* is the rank of individual *i* in the distribution of total deprivation (all domains incldued) in the population, let us call i^j the rank of individual *i* in the distribution of deprivations in domain *j*. It is then easy to check (see, Berrebi and Silber, 1987) that $\left[\sum_{i=1}^{n} b_{ij}\left(\frac{n-2i^j+1}{n}\right)\right]$ represents the Gini index G_j of the deprivations in domain *j* while $\left[\sum_{i=1}^{n} b_{ij}\left(\frac{n-2i+1}{n}\right)\right]$ is called the Pseudo-Gini $PG_j$of the deprivations in domain *j* (See, Fei et al., 1979).

We may therefore rewrite (C-6) as

$$MDI = \bar{d}\{1 + \sum_{j=1}^{J} s_j [PG_j]\} = \bar{d}\left\{1 + \sum_{j=1}^{J} s_j \left[G_j\left(\frac{PG_j}{G_j}\right)\right]\right\} = \bar{d}\{1 + \sum_{j=1}^{J} s_j [G_j(GC_j)]\} \qquad \text{(C-7)}$$

where $GC_j = \left(\frac{PG_j}{G_j}\right)$ is called the Gini correlation coefficient (see, Yitzhaki and Schechtman, 2013).

In other words a domain *j* of deprivation contributes more to the MDI

- the higher the share s_j of domain *j* in the total amount of deprivation in the population (all domains included).
- the higher the Gini index G_j of the deprivations in domain *j*
- the higher the Gini correlation coefficient GC_j for domain *j* (the higher the correlation between the distribution of the deprivations in domain *j* and the distributioon of the total deprivations, all domains included, in the population, this correlation being measured not via the Pearson correlation coefficient but via the Gini correlation coefficient).

<u>Working with mean differences</u>

Let us first recall that $\left[\sum_{i=1}^{n} b_{ij}\left(\frac{n-2i^j+1}{n}\right)\right]$ represents the Gini index G_j of the deprivations in domain *j* while $\left[\sum_{i=1}^{n} b_{ij}\left(\frac{n-2i+1}{n}\right)\right]$ is called the Pseudo-Gini $PG_j$of the deprivations in domain *j*.

Moreover, as already mentioned by Kendall and Stuart (XX), we know that the Gini index is equal to half the ratio of the mean difference over the corresponding mean. We may therefore write that

$$G_j = \left(\frac{1}{2}\right)\left(\frac{\Delta_j}{(\sum_{i=1}^{n} d_{ij})/n}\right) \qquad \text{(C-12)}$$

where Δ_j is the mean difference of the deprivations d_{ij} (within the deprivation domain *j*).

We can similarly define a "Pseudo Mean Difference" and write that

$$PG_j = \left(\frac{1}{2}\right)\left(\frac{P\Delta_j}{(\sum_{i=1}^{n} d_{ij})/n}\right) \quad \text{(C-13)}$$

where $P\Delta_j$ refers to the mean difference of the deprivations d_{ij} (within the deprivation domain *j*).

Combining (D-11), (D-12) and (D-13) we derive that

$$MDI = \left\{\left(\frac{1}{nJ}\right)\sum_{j=1}^{J}\left[\sum_{i=1}^{n} d_{ij}\right]\right\} + \left\{\bar{d}\left[\sum_{j=1}^{J} s_j PG_j\right]\right\}$$

$$=\left\{\left(\frac{1}{J}\right)\sum_{j=1}^{J}\left[\frac{\sum_{i=1}^{n} d_{ij}}{n}\right]\right\} + \left\{\left(\frac{1}{J}\right)\left(\frac{\sum_{j=1}^{J}\sum_{i=1}^{n} d_{ij}}{n}\right)\left[\sum_{j=}^{J}\left(\frac{(\sum_{i=1}^{n} d_{ij})/n}{(\sum_{j=1}^{J}\sum_{i=1}^{n} d_{ij})/n}\right)\left(\frac{1}{2}\right)\left(\frac{P\Delta_j}{(\sum_{i=1}^{n} d_{ij})/n}\right)\right]\right\}$$

$$=\bar{d} + \left(\frac{1}{2}\right)\left(\frac{1}{J}\right)\sum_{j=1}^{J} P\Delta_j = \bar{d} + \left(\frac{1}{2}\right)\left(\frac{1}{J}\right)\sum_{j=1}^{J} \Delta_j\left(\frac{P\Delta_j}{\Delta_j}\right) = \bar{d} + \left(\frac{1}{2}\right)\left(\frac{1}{J}\right)\sum_{j=1}^{J} \Delta_j GC_j \quad \text{(C-14)}$$